Energy in a mechanical wave*

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Abstract

Calculate the energy in a wave on a string.

1 Energy Transport

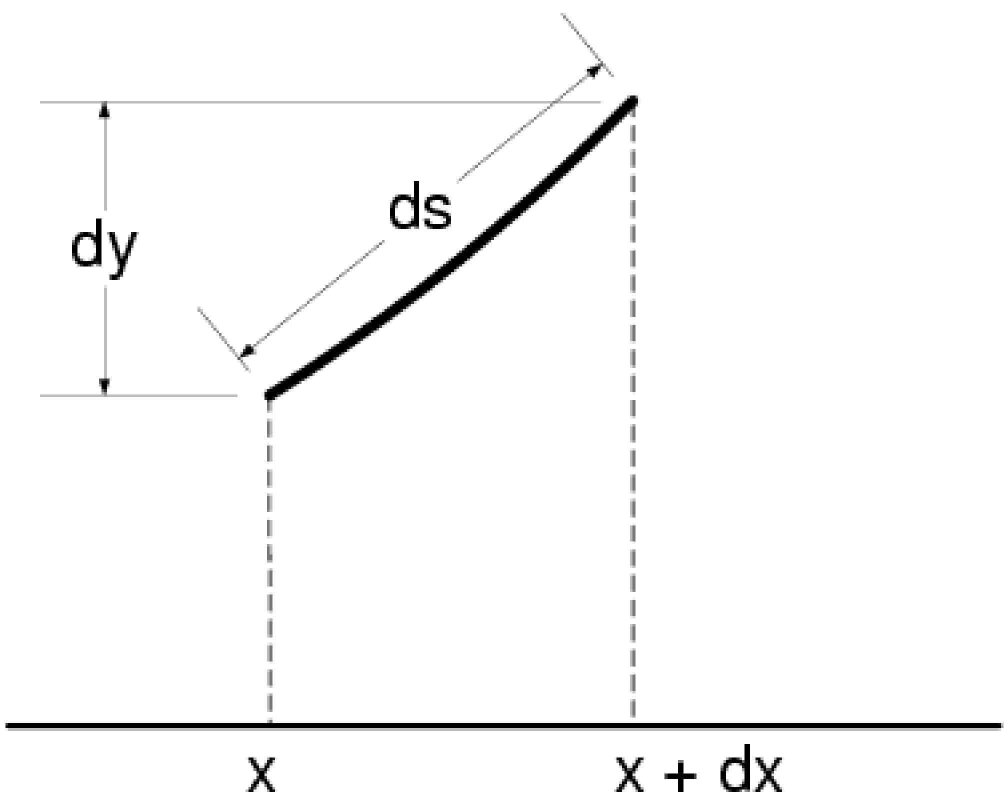

Figure 1

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Lets calculate the energy in a wave of a string:

Consider a fragment of string so small it can be considered straight, as is shown in the figure

The the kinetic energy is $K = \frac{1}{2}m\mathrm{v}^2$ for the string fragment $m = \mu dx$

NOTE: Why is this dx and not ds? Lets consider that the string is not perturbed, then it is horizontal and has mass as given. When the string is perturbed it stretches a little bit - but the mass does not increase.

So we have

$$K = \frac{1}{2}\mu dx\left(\frac{\partial y}{\partial t}\right)^2$$

and using this we can define the energy per unit length, ie. the kinetic energy density:

$$\frac{dK}{dx} = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2$$

When the string segment is stretched from the length dx to the length ds an amount of work $= T\,(ds - dx)$ is done. This is equal to the potential energy stored in the stretched string segment. So the potential energy in this case is:

$$U = T\,(ds - dx)$$

Now

$$\begin{aligned} ds &= \left(dx^2 + dy^2\right)^{1/2} \\ &= dx\left[1 + \left(\tfrac{\partial y}{\partial x}\right)^2\right]^{1/2} \end{aligned}$$

Recall the binomial expansion

$$(1 + A)^n = 1 + nA + \frac{n\,(n-1)\,A^2}{2!} + \frac{n\,(n-1)\,(n-2)\,A^3}{3!} + \dots$$

so

$$ds \approx dx + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx$$

$$U = T\,(ds - dx) \approx \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 dx$$

or the potential energy density

$$\frac{dU}{dx} = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2$$

To get the kinetic energy in a wavelength, lets start with

$$y = A\sin\left(\frac{2\pi x}{\lambda} - \omega t\right)$$

$$\frac{\partial y}{\partial t} = -\omega A\cos\left(\frac{2\pi x}{\lambda} - \omega t\right)$$

Lets evaluate it at time 0.

$$\frac{\partial y}{\partial t}\Big|_{t=0} = -\omega A\cos\left(\frac{2\pi x}{\lambda}\right)$$

so

$$\frac{dK}{dx} = \frac{1}{2}\mu\omega^2A^2\cos^2\left(\frac{2\pi x}{\lambda}\right)$$

now integrate

$$K = \int_0^\lambda \frac{dK}{dx} dx$$
$$= \frac{1}{2}\mu\omega^2 A^2 \int_0^\lambda \cos^2\left(\frac{2\pi x}{\lambda}\right) dx$$

In order to do this integral we use the following trig identity:

$$\cos^2 A = \frac{\cos 2A + 1}{2}$$

so we get

$$K = \frac{1}{2}\mu\omega^2 A^2 \left[\frac{x}{2} + \frac{\lambda}{8\pi}\sin\frac{4\pi x}{\lambda}\right] |_{x=0}^{\lambda}$$
$$= \frac{1}{4}\mu\lambda\omega^2 A^2$$

In similar fashion the potential energy can be found to be

$$U = \frac{1}{4}\mu\lambda\omega^2 A^2.$$

Deriving this will be assigned as a homework problem

So

$$E = K + U = \frac{1}{2}\mu\lambda\omega^2 A^2$$

Power

$$P = \frac{\Delta E}{\Delta t} = \frac{\frac{1}{2}\mu\lambda\omega^2 A^2}{\tau}$$
$$= \frac{1}{2}\mu\omega^2 A^2 \mathrm{v}$$

Where I have used $\tau = 1/\nu$ and $\lambda\nu = \mathrm{v}$ thus $\tau = \lambda/\mathrm{v}$