# Diffraction from a Circular Aperture*

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**Abstract**

We examine diffraction through a circular aperture.

## 1 Circular Aperture

The circular aperture is particularly important because it is used a lot in optics. A telescope typically has a circular aperture for example.

We can use the same expression for the E field that we had for the rectangular aperture for any possible aperture, as long as the limits of integration are appropriate. So we can write

$$\vec{E}=\frac{\vec{\varepsilon}_A}{R}e^{i(kR-\omega t)}\int\int_{aperture}e^{-iK(Yy+Zz)/R}dydz$$

For a circular aperture this integration is most easily done with cylindrical coordinates. Look at the figure

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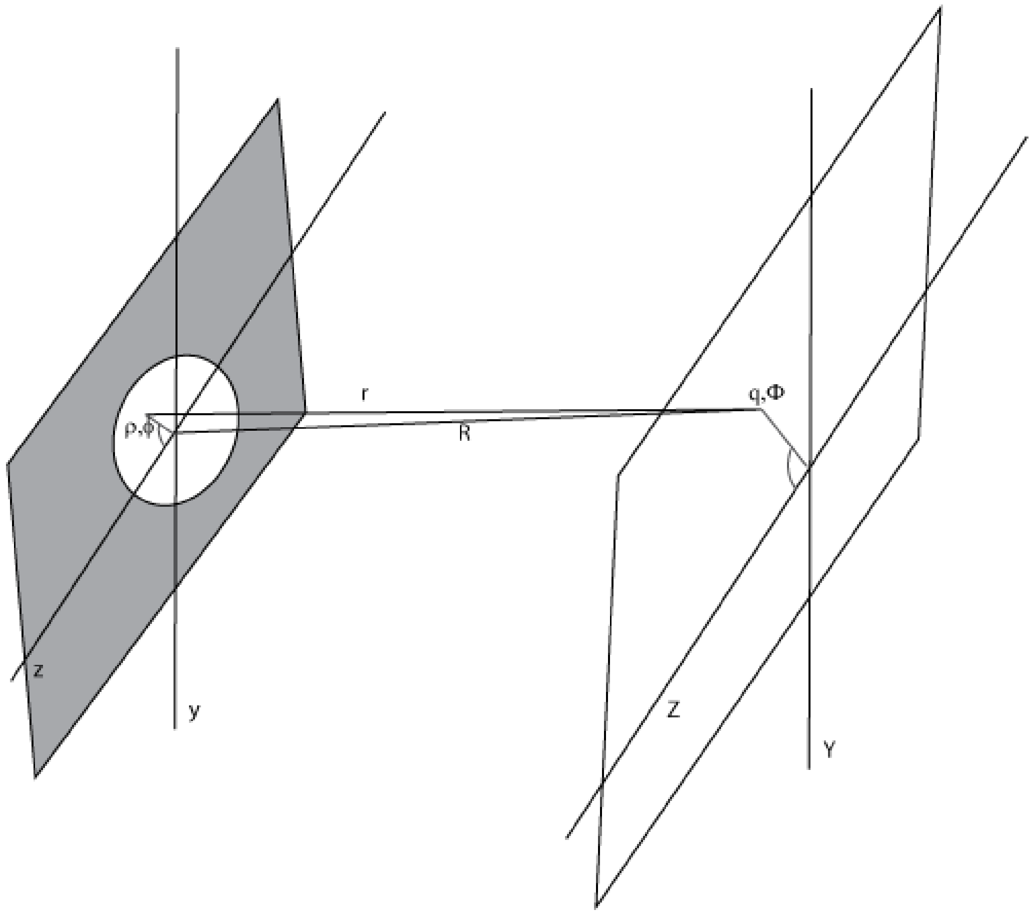


**Figure 1**

Then we have

$$z = \rho \cos\phi$$

$$y = \rho \sin\phi$$

$$Z = q \cos\Phi$$

$$Y = q \sin\Phi$$

Then

$$Yy + Zz = \rho q \cos\phi \cos\Phi + \rho q \sin\phi \sin\Phi$$

or

$$Yy + Zz = \rho q \cos\left(\phi - \Phi\right)$$

and the integral becomes

$$\vec{E}=\frac{\vec{\varepsilon}_A}{R}e^{i(kR-\omega t)}\int_0^a\int_0^{2\pi}e^{-iK\rho q\cos(\phi-\Phi)/R}\rho d\rho d\phi$$

In order to do this integral we need to learn a little about Bessel functions.

$$J_0(u)=\frac{1}{2\pi}\int_0^{2\pi}e^{iu\cos v}dv$$

Is the definition of a Bessel function of the first kind order 0.

$$J_m(u)=\frac{1}{2\pi}\int_0^{2\pi}e^{i(mv+u\cos v)}dv$$

Is the definition of a Bessel function of the first kind order m.

They have a number of interesting properties such as the recurrence relations

$$\frac{d}{du}\left[u^m J_m(u)\right)=u^m J_{m-1}(u)$$

so that for example when $m=1$

$$\int_0^u u' J_0(u')\,du'=uJ_1(u).$$

In order to numerically calculate the value of a Bessel function one uses the expansion

$$J_m(x)=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!\,(m+s)!}\left(\frac{x}{2}\right)^{m+2s}.$$

Now we want to evaluate the integral

$$\vec{E}=\frac{\vec{\varepsilon}_A}{R}e^{i(kR-\omega t)}\int_0^a\int_0^{2\pi}e^{-iK\rho q\cos(\phi-\Phi)/R}\rho d\rho d\phi$$

which we can do at any value of $\Phi$ since the problem is symmetric about $\Phi$. So we can simplify things greatly if we do the integral at $\Phi=0$

$$\vec{E}=\frac{\vec{\varepsilon}_A}{R}e^{i(kR-\omega t)}\int_0^a\int_0^{2\pi}e^{-iK\rho q\cos(\phi)/R}\rho d\rho d\phi$$

which becomes

$$\vec{E}=\frac{\vec{\varepsilon}_A}{R}e^{i(kR-\omega t)}2\pi\int_0^a J_0(-K\rho q/R)\,\rho d\rho$$

Now $J_0$ is an even function so we can drop the minus sign and rewrite the expression as

$$\vec{E}=\frac{\vec{\varepsilon}_A}{R}e^{i(kR-\omega t)}2\pi\int_0^a J_0(K\rho q/R)\,\rho d\rho$$

To do this integral we change variables

$$w=k\rho q/R$$

$$\rho=\frac{wR}{kq}$$

$$d\rho = \frac{R}{kq}dw$$

so that

$$\begin{aligned}\int_0^a J_0(K\rho q/R)\,\rho d\rho &= \int_0^{kaq/R}\left(\frac{R}{kq}\right)^2 J_0(w)\,wdw \\ &= \left(\frac{R}{kq}\right)^2\left(\frac{kaq}{R}\right)J_1(kaq/R) \\ &= a^2\left(\frac{R}{kaq}\right)J_1(kaq/R) \\ &= a^2\frac{J_1(kaq/R)}{kaq/R}\end{aligned}$$

So finally we have the result

$$\vec{E}=\vec{\varepsilon_A}\;\frac{e^{i(kR-\omega t)}}{R}2\pi a^2\frac{J_1(kaq/R)}{kaq/R}$$

Or recognizing that $\pi a^2$ is the area of the aperture $A$ and squaring to get the intensity we write

$$I = I_0\left[\frac{2J_1(kaq/R)}{kaq/R}\right]^2$$

If you want to write this in terms of the angle $\theta$ then one uses the fact that $q/R = \sin\theta$

$$I(\theta) = I(0)\left[\frac{2J_1(ka\sin\theta)}{ka\sin\theta}\right]^2$$

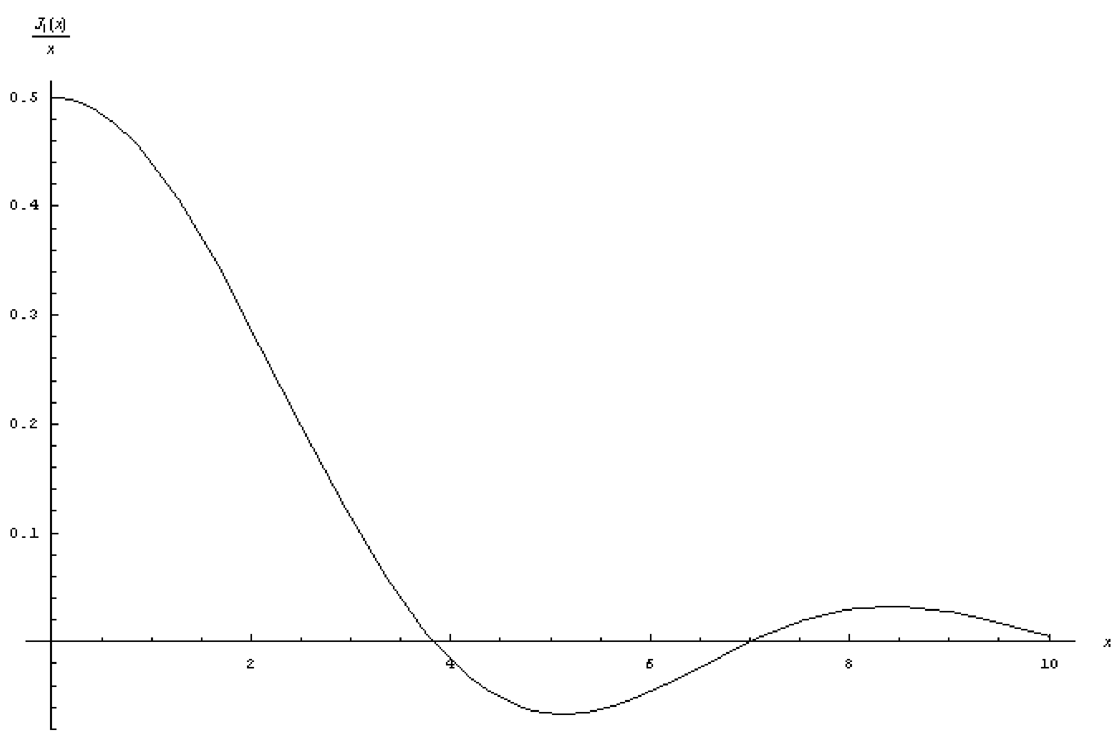


**Figure 2**

Above is a plot of the function $J_1(x)/x$. Notice how it peaks at 1/2 which is why there is the factor of two in the expression for the irradiance. Below is a 3D plot of the same thing (ie. $J_1(r)/r$). Notice the rings.

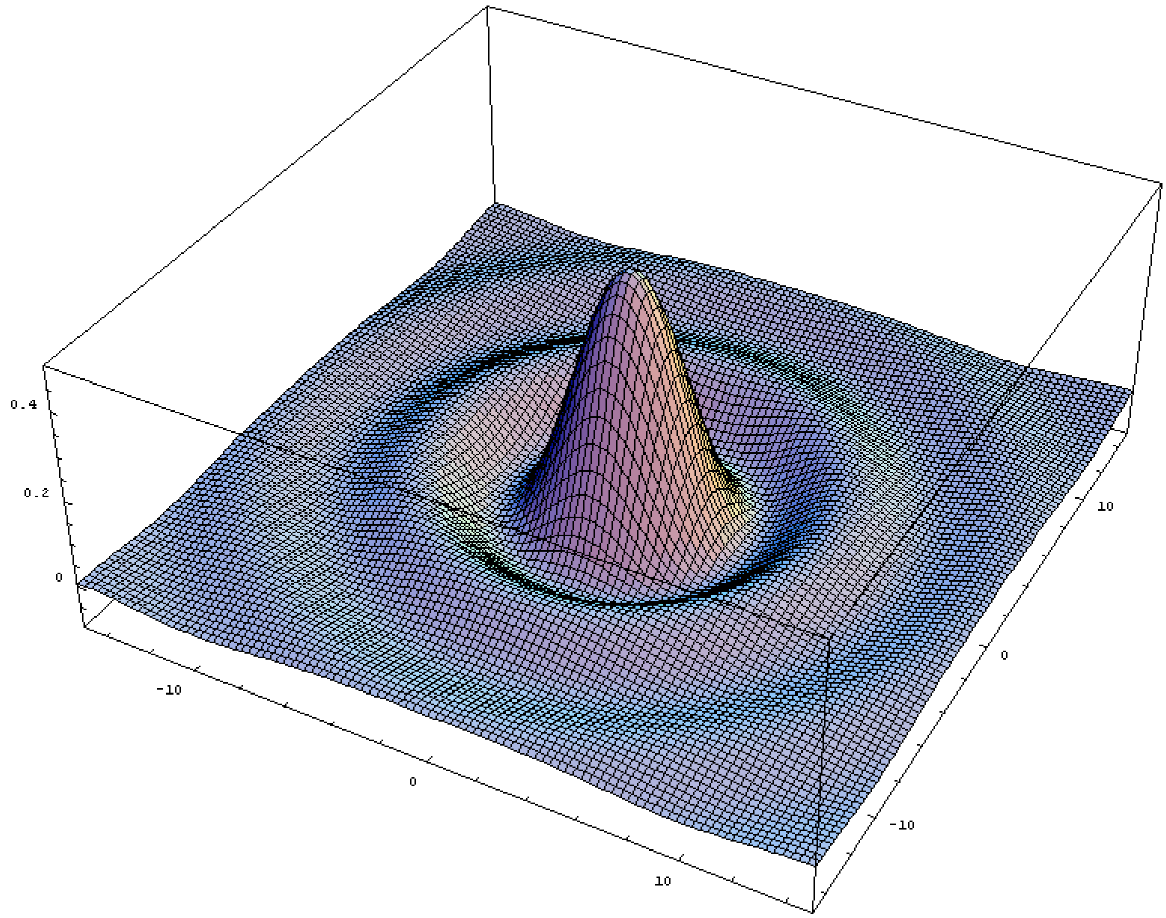


**Figure 3**

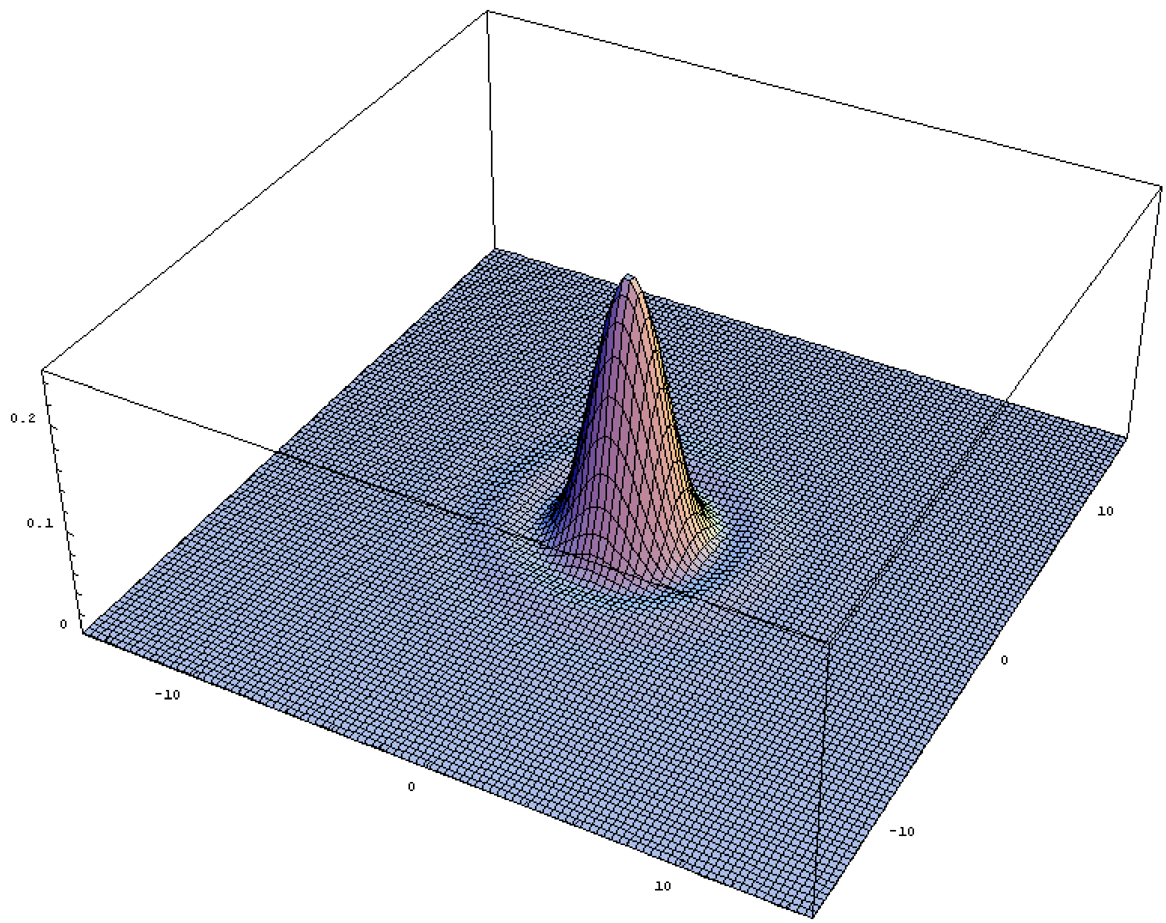


**Figure 4**

Above is a plot of $(J_1(r)/r)^2$ which corresponds to the irradiance one sees. The central peak out to the first ring of zero is called the Airy disk. This occurs at $J_1(r)/r = 0$ which can be numerically evaluated to give $r = 3.83$ for the first ring.

For our circular aperture above this means the first zero occurs at

$$kaq_1/R = 3.83$$

or

$$\frac{2\pi}{\lambda}\frac{aq_1}{R} = 3.83$$

$$q_1 = \frac{1.22R\lambda}{2a}$$

In our case $a$ is the radius of the aperture and we can rewrite the expression using the diameter $D = 2a$

$$q_1 = 1.22\lambda R/D$$

Light passing through any circular aperture is going to be diffracted in this manner and this sets the limit of resolution on an optical device such as a telescope. Say one is trying resolve two sources, we can say the limit of resolution is when the central spot of one Airy disk is on the zero of the other Airy disk. This is known as the Raleigh critereon. While it is possible to define other crtieria, this is the most commenly used. See for example the plots below

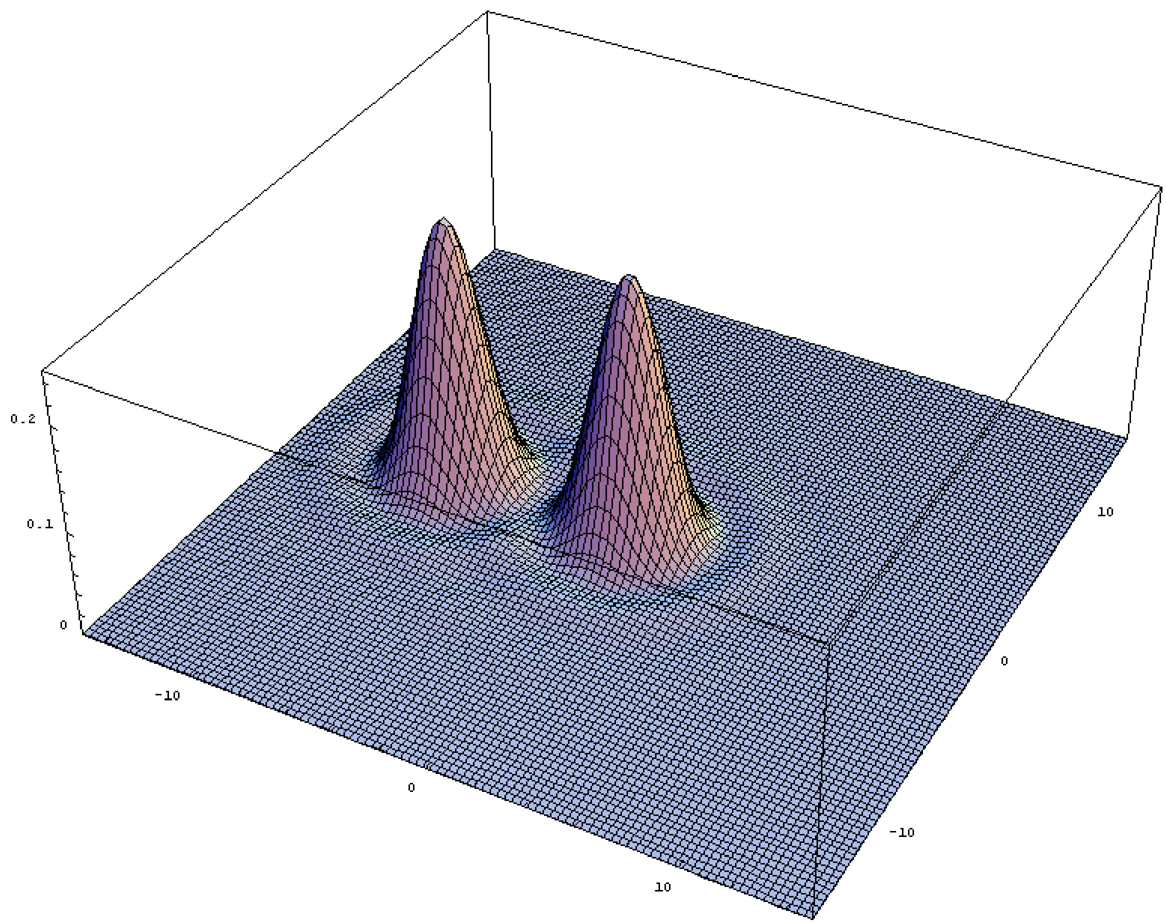


**Figure 5**

In the above plot, the two sources can clearly be resolved. In the plot below, the two sources are going to be difficult to resolve.

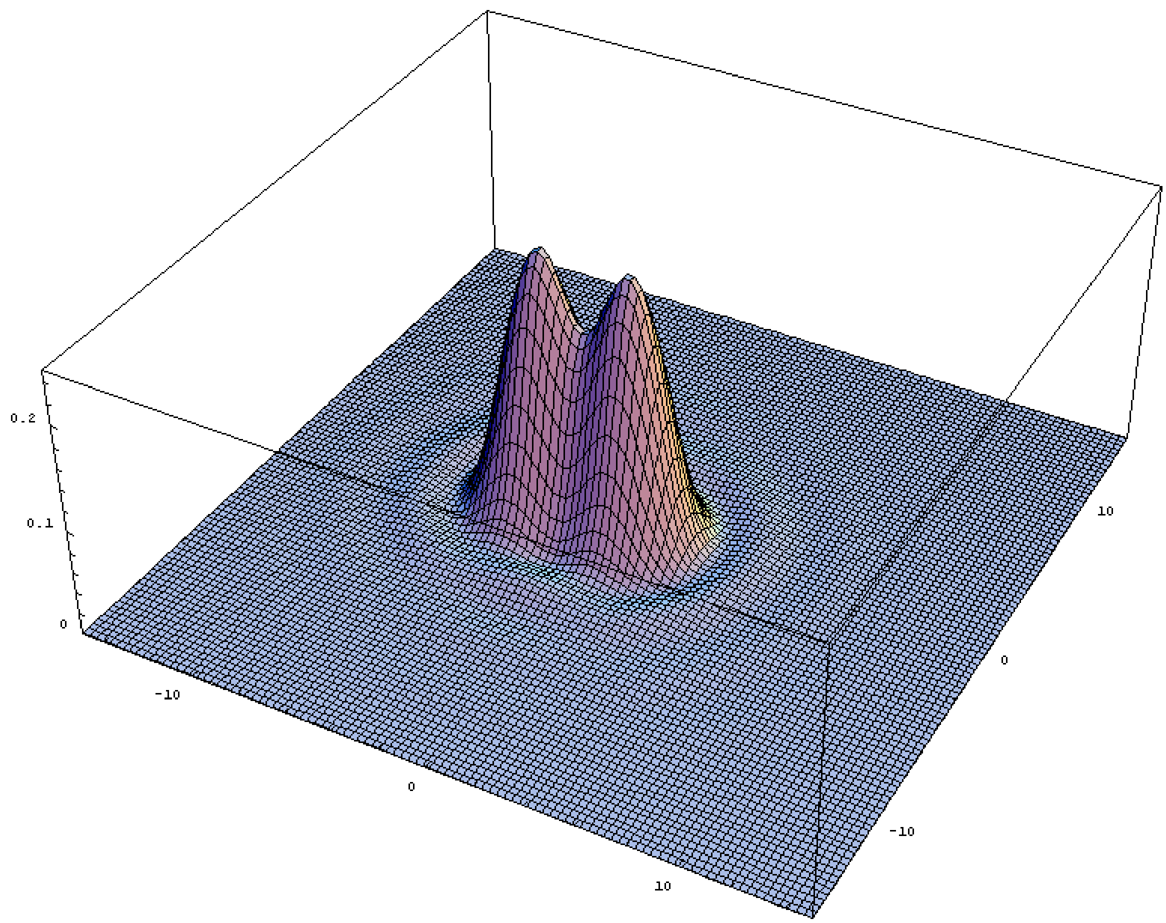


**Figure 6**

So we say that the limit of our resolution occurs when the distance $\Delta q$ between two sources is

$$\Delta q = 1.22 R\lambda / D$$

or in the small angle limit $\Delta\theta = \Delta q / R$

$$\Delta\theta = 1.22\lambda / D$$