THE MEAN, VARIANCE, AND STANDARD DEVIATION*

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Abstract

This course is a short series of lectures on Introductory Statistics. Topics covered are listed in the Table of Contents. The notes were prepared by Ewa Paszek and Marek Kimmel. The development of this course has been supported by NSF 0203396 grant.

1 The MEAN, VARIANCE, and STANDARD DEVIATION

1.1 MEAN and VARIANCE

Certain mathematical expectations are so important that they have special names. In this section we consider two of them: the mean and the variance.

1.1.1

Mean Value

If $\mathbf{X}$ is a random variable with p.d.f. $f(x)$ of the discrete type and space $\mathbf{R}$=$(b_1, b_2, b_3, ...)$, then $E(X) = \sum_R xf(x) = b_1 f(b_1) + b_2 f(b_2) + b_3 f(b_3) + ...$ is the weighted average of the numbers belonging to $\mathbf{R}$, where the weights are given by the p.d.f. $f(x)$.

We call $E(X)$**the mean** of $\mathbf{X}$ (or **the mean of the distribution**) and denote it by μ. That is, $\mu = E(X)$.

NOTE: In mechanics, the weighted average of the points $b_1, b_2, b_3, ...$ in one-dimensional space is called the centroid of the system. Those without the mechanics background can think of the centroid as being the point of balance for the system in which the weights $f(b_1), f(b_2), f(b_3), ...$ are places upon the points $b_1, b_2, b_3, ...$

Example 1

Let $\mathbf{X}$ have the p.d.f.

$$f(x) = \{ \begin{array}{l} \frac{1}{8}, x = 0, 3, \\ \frac{3}{8}, x = 1, 2. \end{array}$$

The mean of $\mathbf{X}$ is

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$$\mu = E\left[X = 0\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right) = \frac{3}{2}.\right.$$

The example below shows that if the outcomes of **X** are equally likely (i.e., each of the outcomes has the same probability), then the mean of **X** is the arithmetic average of these outcomes.

Example 2

Roll a fair die and let **X** denote the outcome. Thus **X** has the p.d.f.

$$f(x) = \frac{1}{6}, x = 1, 2, 3, 4, 5, 6.$$

Then,

$$E(X) = \sum_{x=1}^{6} x\left(\frac{1}{6}\right) = \frac{1+2+3+4+5+6}{6} = \frac{7}{2},$$

which is the arithmetic average of the first six positive integers.

1.1.2

Variance

It was denoted that the mean $\mu = E(X)$ is the centroid of a system of weights of measure of the central location of the probability distribution of **X**. **A measure of the dispersion or spread of a distribution is defined as follows:**

If $u(x) = (x-\mu)^2$ and $E\left[(X-\mu)^2\right]$ exists, **the variance**, frequently denoted by σ^2 or $Var(X)$, of a random variable **X** of the discrete type (or variance of the distribution) is defined by

$$\sigma^2 = E\left[(X-\mu)^2\right] = \sum_R (x-\mu)^2 f(x). \tag{1}$$

The positive square root of the variance is called **the standard deviation of X** and is denoted by

$$\sigma = \sqrt{Var(X)} = \sqrt{E\left[(X-\mu)^2\right]}. \tag{2}$$

Example 3

Let the p.d.f. of **X** by defined by

$$f(x) = \frac{x}{6}, x = 1, 2, 3.$$

The mean of **X** is

$$\mu = E(X) = 1\left(\frac{1}{6}\right) + 2\left(\frac{2}{6}\right) + 3\left(\frac{3}{6}\right) = \frac{7}{3}.$$

To find the variance and standard deviation of **X** we first find

$$E\left(X^2\right) = 1^2\left(\frac{1}{6}\right) + 2^2\left(\frac{2}{6}\right) + 3^2\left(\frac{3}{6}\right) = \frac{36}{6} = 6.$$

Thus the variance of **X** is

$$\sigma^2 = E\left(X^2\right) - \mu^2 = 6 - \left(\frac{7}{3}\right)^2 = \frac{5}{9},$$

and the standard deviation of **X** is

Example 4
Let **X** be a random variable with mean μ_x and variance σ_x^2. Of course, $Y = aX + b$, where a and b are constants, is a random variable, too. The mean of **Y** is

$$\mu_Y = E(Y) = E(aX + b) = aE(X) + b = a\mu_X + b.$$

Moreover, the variance of **Y** is

$$\sigma_Y^2 = E\left[(Y - \mu_Y)^2\right] = E\left[(aX + b - a\mu_X - b)^2\right] = E\left[a^2(X - \mu_X)^2\right] = a^2\sigma_X^2.$$

1.1.3

Moments of the distribution
Let **r** be a positive integer. If

$$E(X^r) = \sum_R x^r f(x)$$

exists, it is called **the rth moment of the distribution** about the origin. The expression moment has its origin in the study of mechanics.

In addition, the expectation

$$E\left[(X - b)^r\right] = \sum_R x^r f(x)$$

is called **the rth moment of the distribution about b**. For a given positive integer r.

$$E\left[(X)_r\right] = E\left[X(X-1)(X-2)\cdots(X-r+1)\right]$$

is called **the rth factorial moment**.

NOTE: The second factorial moment is equal to the difference of the second and first moments:

$$E\left[X(X-1)\right] = E\left(X^2\right) - E(X).$$

There is another formula that can be used for computing the variance that uses the second factorial moment and sometimes simplifies the calculations.

First find the values of $E(X)$ and $E\left[X(X-1)\right]$. Then

$$\sigma^2 = E\left[X(X-1)\right] + E(X) - \left[E(X)\right]^2,$$

since using the distributive property of **E**, this becomes

$$\sigma^2 = E\left(X^2\right) - E(X) + E(X) - \left[E(X)\right]^2 = E\left(X^2\right) - \mu^2.$$

Example 5
Let continue with example 4 (Example 4), it can be find that

$$E\left[X\left(X-1\right)\right]=1\left(0\right)\left(\frac{1}{6}\right)+2\left(1\right)\left(\frac{2}{6}\right)+3\left(2\right)\left(\frac{3}{6}\right)=\frac{22}{6}.$$

Thus

$$\sigma^2=E\left[X\left(X-1\right)\right]+E\left(X\right)-\left[E\left(X\right)\right]^2=\frac{22}{6}+\frac{7}{3}-\left(\frac{7}{3}\right)^2=\frac{5}{9}.$$

NOTE: Recall the empirical distribution is defined by placing the weight (probability) of $1/\mathbf{n}$ on each of $\mathbf{n}$ observations $x_1, x_2, ..., x_n$. Then the mean of this empirical distribution is

$$\sum_{i=1}^{n} x_i \frac{1}{n}=\frac{\sum_{i=1}^{n} x_i}{n}=\overline{x}.$$

The symbol $\overline{x}$ represents **the mean of the empirical distribution**. It is seen that $\overline{x}$ is usually close in value to $\mu=E\left(X\right)$; thus, when μ is unknown, $\overline{x}$ will be used to estimate μ.

Similarly, **the variance of the empirical distribution** can be computed. Let $\mathbf{v}$ denote this variance so that it is equal to

$$v=\sum_{i=1}^{n}\left(x_i-\overline{x}\right)^2\frac{1}{n}=\sum_{i=1}^{n} x_i^2\frac{1}{n}-\overline{x}^2=\frac{1}{n}\sum_{i=1}^{n} x_i^2-\overline{x}^2.$$

This last statement is true because, in general,

$$\sigma^2=E\left(X^2\right)-\mu^2.$$

NOTE: There is a relationship between the sample variance s^2 and variance $\mathbf{v}$ of the empirical distribution, namely $s^2=ns/\left(n-1\right)$. Of course, with large $\mathbf{n}$, the difference between s^2 and $\mathbf{v}$ is very small. Usually, we use s^2 to estimate σ^2 when σ^2 is unknown.

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2.1

NOTE: BERNOULLI TRIALS and BINOMIAL DISTRIBUTION[1]

[1] "BERNOULLI TRIALS and the BINOMIAL DISTRIBUTION": Section BERNOULLI TRIALS AND THE BINOMIAL DISTRIBUTION <http://cnx.org/content/m13123/latest/#sec_1>