POISSON DISTRIBUTION[*]

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Abstract

This course is a short series of lectures on Introductory Statistics. Topics covered are listed in the Table of Contents. The notes were prepared by Ewa Paszek and Marek Kimmel. The development of this course has been supported by NSF 0203396 grant.

1 POISSON DISTRIBUTION

Some experiments results in counting the number of times particular events occur in given times of on given physical objects. For example, we would count the number of phone calls arriving at a switch board between 9 and 10 am, the number of flaws in 100 feet of wire, the number of customers that arrive at a ticket window between 12 noon and 2 pm, or the number of defects in a 100-foot roll of aluminum screen that is 2 feet wide. Each count can be looked upon as a random variable associated with an approximate Poisson process provided the conditions in the definition below are satisfied.

Definition 1: POISSON PROCCESS

Let the number of changes that occur in a given continuous interval be counted. We have **an approximate Poisson process** with parameter $\lambda > 0$ if the following are satisfied:

1. The number of changes occurring in nonoverlapping intervals are independent.
2. The probability of exactly one change in a sufficiently short interval of length **h** is approximately λh .
3. The probability of two or more changes in a sufficiently short interval is essentially zero.

1.1

Suppose that an experiment satisfies the three points of an approximate Poisson process. Let **X** denote the number of changes in an interval of "length 1" (where "length 1" represents one unit of the quantity under consideration). We would like to find an approximation for $P(X = x)$, where **x** is a nonnegative integer. To achieve this, we partition the unit interval into **n** subintervals of equal length $1/n$. If **N** is sufficiently large (i.e., much larger than **x**), one shall approximate the probability that **x** changes occur in this unit interval by finding the probability that one change occurs exactly in each of exactly **x** of these **n** subintervals. The probability of one change occurring in any one subinterval of length $1/n$ is approximately $\lambda(1/n)$ by condition (2). The probability of two or more changes in any one subinterval is essentially zero by condition (3). So for each subinterval, exactly one change occurs with a probability of approximately $\lambda(1/n)$. Consider the occurrence or nonoccurrence of a change in each subinterval as a Bernoulli trial. By

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condition (1) we have a sequence of **n** Bernoulli trials with probability **p** approximately equal to $\lambda (1/n)$. Thus an approximation for $P (X = x)$ is given by the binomial probability

$$\frac{n!}{x! (n - x)!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}.$$

In order to obtain a better approximation, choose a large value for **n**. If **n** increases without bound, we have that

$$\lim_{n\to\infty} \frac{n!}{x! (n - x)!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x} = \lim_{n\to\infty} \frac{n (n - 1) \dots (n - x + 1)}{n^x} \frac{\lambda^x}{x!} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-x}.$$

Now, for fixed **x**, we have

$$\lim_{n\to\infty} \frac{n(n-1)\dots(n-x+1)}{n^x} = \lim_{n\to\infty} \left[1 \left(1 - \frac{1}{n}\right) \dots \left(1 - \frac{x-1}{n}\right)\right] = 1,$$
$$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda},$$

and

$$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{-x} = 1.$$

Thus,

$$\lim_{n\to\infty} \frac{n!}{x! (n - x)!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x} = \frac{\lambda^x e^{-\lambda}}{x!} = P (X = x),$$

approximately. The distribution of probability associated with this process has a special name.

1.1.1

Definition 2: POISSON DISTRIBUTION

We say that the random variable **X** has **a Poisson distribution** if its p.d.f. is of the form

$$f (x) = \frac{\lambda^x e^{-\lambda}}{x!}, x = 0, 1, 2, \dots,$$

where $\lambda > 0$.

It is easy to see that $f (x)$ enjoys the properties pf a p.d.f. because clearly $f (x) \geq 0$ and, from the Maclaurin's series expansion of e^λ , we have

$$\sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda} e^\lambda = 1.$$

To discover the exact role of the parameter $\lambda > 0$, let us find some of the characteristics of the Poisson distribution . The mean for the Poisson distribution is given by

$$E (X) = \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=1}^{\infty} x \frac{\lambda^x}{(x - 1)!},$$

because (0) $f (0) = 0$ and $x/x! = 1/ (x - 1)!$, when $x > 0$.

If we let $k = x - 1$, then

$$E (X) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{k!} = \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \lambda e^{-\lambda} e^\lambda = \lambda.$$

That is, **the parameter λ is the mean of the Poisson distribution**. On the Figure 1 (Figure 1: Poisson Distribution) is shown the p.d.f. and c.d.f. of the Poisson Distribution for $\lambda = 1, \lambda = 4, \lambda = 10$.
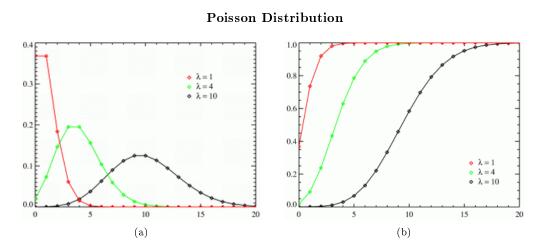
Poisson Distribution

(a)   (b)

Figure 1:   The p.d.f. and c.d.f. of the Poisson Distribution for $\lambda = 1, \lambda = 4, \lambda = 10$. (a) The p.d.f. function. (b) The c.d.f. function.

To find the variance, we first determine the second factorial moment $E\left[X\left(X-1\right)\right]$. We have,

$$E\left[X\left(X-1\right)\right] = \sum_{x=0}^{\infty} x\left(x-1\right)\frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda}\sum_{x=2}^{\infty}\frac{\lambda^x}{(x-2)!},$$

because $(0)(0-1)f(0) = 0, (1)(1-1)f(1) = 0$, and $x\left(x-1\right)/x! = 1/\left(x-2\right)!$, when $x > 1$.
If we let $k = x - 2$, then

$$E\left[X\left(X-1\right)\right] = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^{k+2}}{k!} = \lambda^2 e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^x}{k!} = \lambda^2 e^{-\lambda}e^{\lambda} = \lambda^2.$$

Thus,

$$Var\left(X\right) = E\left(X^2\right) - \left[E\left(X\right)\right]^2 = E\left[X\left(X-1\right)\right] + E\left(X\right) - \left[E\left(X\right)\right]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

That is, for the Poisson distribution, $\mu = \sigma^2 = \lambda$.

1.1.2

Example 1
Let **X** have a Poisson distribution with a mean of $\lambda = 5$, (it is possible to use the tabularized Poisson distribution).

$$P\left(X \leq 6\right) = \sum_{x=0}^{6}\frac{5^x e^{-5}}{x!} = 0.762,$$
$$P\left(X > 5\right) = 1 - P\left(X \leq 5\right) = 1 - 0.616 = 0.384,$$

and
$$P(X = 6) = P(X \leq 6) - P(X \leq 5) = 0.762 - 0.616 = 0.146.$$

Example 2

Telephone calls enter a college switchboard on the average of two every 3 minutes. If one assumes an approximate Poisson process, what is the probability of five or more calls arriving in a 9-minute period? Let **X** denotes the number of calls in a 9-minute period. We see that $E(X) = 6$; that is, on the average, sic calls will arrive during a 9-minute period. Thus using tabularized data,

$$P(X \geq 5) = 1 - P(X \leq 4) = 1 - \sum_{x=0}^{4} \frac{6^x e^{-6}}{x!} = 1 - 0.285 = 0.715.$$

1.1.3

NOTE: Not only is the Poisson distribution important in its own right, but it can also be used to approximate probabilities for a binomial distribution.

If **X** has a Poisson distribution with parameter λ , we saw that with **n** large,

$$P(X = x) \approx \begin{pmatrix} n \\ x \end{pmatrix} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x},$$

where, $p = \lambda/n$ so that $\lambda = np$ in the above binomial probability. That is, if **X** has the binomial distribution $b(n,p)$ with large **n**, then

$$\frac{(np)^x e^{-np}}{x!} = \begin{pmatrix} n \\ x \end{pmatrix} p^x (1-p)^{n-x}.$$

This approximation is reasonably good if **n** is large. But since λ was fixed constant in that earlier argument, **p** should be small since $np = \lambda$. In particular, the approximation is quite accurate if $n \geq 20$ and $p \leq 0.05$, and it is very good if $n \geq 100$ and $np \leq 10$.

1.1.4

Example 3

A manufacturer of Christmas tree bulbs knows that 2% of its bulbs are defective. Approximate the probability that a box of 100 of these bulbs contains at most three defective bulbs. Assuming independence, we have binomial distribution with parameters **p**=0.02 and **n**=100. The Poisson distribution with $\lambda = 100(0.02) = 2$ gives

$$\sum_{x=0}^{3} \frac{2^x e^{-2}}{x!} = 0.857,$$

using the binomial distribution, we obtain, after some tedious calculations,

$$\sum_{x=0}^{3} \begin{pmatrix} 100 \\ x \end{pmatrix} (0.02)^x (0.98)^{100-x} = 0.859.$$

Hence, in this case, the Poisson approximation is extremely close to the true value, but much easier to find.