NORMAL DISTRIBUTION*

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Abstract

This course is a short series of lectures on Introductory Statistics. Topics covered are listed in the Table of Contents. The notes were prepared by Ewa Paszek and Marek Kimmel. The development of this course has been supported by NSF 0203396 grant.

1 NORMAL DISTRIBUTION

The normal distribution is perhaps the most important distribution in statistical applications since many measurements have (approximate) normal distributions. One explanation of this fact is the role of the normal distribution in the Central Theorem.

Definition 1:
1. The random variable **X** has a normal distribution if its p.d.f. is defined by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], -\infty < x < \infty, \tag{1}$$

where μ and σ^2 are parameters satisfying $-\infty < \mu < \infty, 0 < \sigma < \infty$, and also where $\exp[v]$ means e^v.
2. Briefly, we say that **X** is $N(\mu, \sigma^2)$

1.1 Proof of the p.d.f. properties

Clearly, $f(x) > 0$. Let now evaluate the integral:

$$I = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx,$$

showing that it is equal to 1. In the integral, change the variables of integration by letting $z = (x-\mu)/\sigma$. Then,

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz,$$

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since $I > 0$, if $I^2 = 1$, then $I = 1$.

Now

$$I^2 = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} e^{-x^2/2}dx\right]\left[\int_{-\infty}^{\infty} e^{-y^2/2}dy\right],$$

or equivalently,

$$I^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{x^2+y^2}{2}\right)dxdy.$$

Letting $x = r\cos\theta, y = r\sin\theta$ (i.e., using polar coordinates), we have

$$I^2 = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}rdrd\theta = \frac{1}{2\pi}\int_0^{2\pi} d\theta = \frac{1}{2\pi}2\pi = 1.$$

1.2

The **mean** and the **variance** of the normal distribution is as follows:

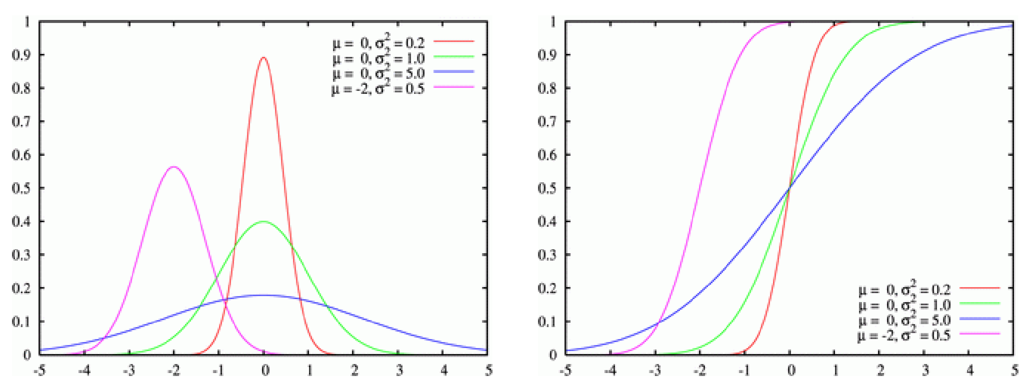
$$E(X) = \mu$$

and

$$Var(X) = \mu^2 + \sigma^2 - \mu^2 = \sigma^2.$$

That is, the parameters μ and σ^2 in the p.d.f. are the mean and the variance of **X**.

Normal Distribution



(a) (b)

Figure 1: p.d.f. and c.d.f graphs of the Normal Distribution (a) Probability Density Function (b) Cumulative Distribution Function

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Example 1

If the p.d.f. of **X** is

$$f\left(x\right)=\frac{1}{\sqrt{32\pi}}\exp\left[-\frac{\left(x+7\right)^{2}}{32}\right],-\infty<x<\infty,$$

then **X** is $N\left(-7,16\right)$

That is, **X** has a normal distribution with a mean μ =-7, variance σ^{2} =16, and the moment generating function

$$M\left(t\right)=\exp\left(-7t+8t^{2}\right).$$