CONFIDENCE INTERVALS I*

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Abstract

This course is a short series of lectures on Introductory Statistics. Topics covered are listed in the Table of Contents. The notes were prepared by Ewa Paszek and Marek Kimmel. The development of this course has been supported by NSF 0203396 grant.

1 CONFIDENCE INTERVALS I

Definition 1:
Given a random sample $X_1, X_2, ..., X_n$ from a normal distribution $N\left(\mu, \sigma^2\right)$, consider the closeness of $\overline{X}$, the unbiased estimator of μ, to the unknown μ. To do this, the error structure (distribution) of $\overline{X}$, namely that $\overline{X}$ is $N\left(\mu, \sigma^2/n\right)$, is used in order to construct what is called **a confidence interval** for the unknown parameter μ, when the variance σ^2 is known.

1.1

For the probability $1-\alpha$, it is possible to find a number $z_{\alpha/2}$, such that

$$P\left(-z_{\alpha/2} \le \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \le z_{\alpha/2}\right) = 1-\alpha.$$

For example, if $1-\alpha = 0.95$, then $z_{\alpha/2} = z_{0.025} = 1.96$ and if $1-\alpha = 0.90$, then $z_{\alpha/2} = z_{0.05} = 1.645$. Recalling that $\sigma > 0$, the following inequalities are equivalent :

$$-z_{\alpha/2} \le \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \le z_{\alpha/2}$$

and

$$-z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right) \le \overline{X}-\mu \le z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right),$$

$$-\overline{X}-z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right) \le -\mu \le -\overline{X}+z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right),$$

$$\overline{X}+z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right) \ge \mu \ge \overline{X}-z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right).$$

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Thus, since the probability of the first of these is 1-1 $-\alpha$, the probability of the last must also be $1-\alpha$, because the latter is true if and only if the former is true. That is,

$$P\left[\overline{X}-z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)\leq\mu\leq-\overline{X}+z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)\right]=1-\alpha.$$

So the probability that the random interval

$$\left[\overline{X}-z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right),\overline{X}+z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)\right]$$

includes the unknown mean μ is $1-\alpha$.

Definition 2:

1. Once the sample is observed and the sample mean computed equal to $\overline{x}$, the interval

$$\overline{x}-z_{\alpha/2}\left(\sigma/\sqrt{n}\right),\overline{x}+z_{\alpha/2}\left(\sigma/\sqrt{n}\right)$$

is a known interval. Since the probability that the random interval covers μ before the sample is drawn is equal to $1-\alpha$, call the computed interval, $\overline{x}\pm z_{\alpha/2}\left(\sigma/\sqrt{n}\right)$(for brevity), a $100\left(1-\alpha\right)\%$**confidence interval** for the unknown mean μ.

2. The number $100\left(1-\alpha\right)\%$, or equivalently, $1-\alpha$, is called **the confidence coefficient**.

1.2

For illustration,

$$\overline{x}\pm 1.96\left(\sigma/\sqrt{n}\right)$$

is a 95% confidence interval for μ.

It can be seen that the confidence interval for μ is centered at the point estimate $\overline{x}$ and is completed by subtracting and adding the quantity $z_{\alpha/2}\left(\sigma/\sqrt{n}\right)$.

NOTE: as **n** increases, $z_{\alpha/2}\left(\sigma/\sqrt{n}\right)$ decreases, resulting **n** a shorter confidence interval with the same confidence coefficient $1-\alpha$

A shorter confidence interval indicates that there is more reliance in $\overline{x}$ as an estimate of μ. For a fixed sample size **n**, the length of the confidence interval can also be shortened by decreasing the confidence coefficient $1-\alpha$. But if this is done, shorter confidence is achieved by losing some confidence.

Example 1

Let $\overline{x}$ be the observed sample mean of 16 items of a random sample from the normal distribution $N\left(\mu,\sigma^2\right)$. A 90% confidence interval for the unknown mean μ is

$$\left[\overline{x}-1.645\sqrt{\frac{23.04}{16}},\overline{x}+1.645\sqrt{\frac{23.04}{16}}\right].$$

For a particular sample this interval either does or does not contain the mean μ. However, if many such intervals were calculated, it should be true that about 90% of them contain the mean μ.

If one cannot assume that the distribution from which the sample arose is normal, one can still obtain an approximate confidence interval for μ . By the Central Limit Theorem the ratio $\left(\overline{X}-\mu\right)/\left(\sigma/\sqrt{n}\right)$ has, provided that **n** is large enough, the approximate normal distribution $N\left(0,1\right)$ when the underlying distribution is not normal. In this case

$$P\left(-z_{\alpha/2}\leq\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\leq z_{\alpha/2}\right)\approx 1-\alpha,$$

and

$$\left[\overline{x}-z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right),\overline{x}+z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)\right]$$

is an approximate $100\,(1-\alpha)\,\%$ confidence interval for μ. The closeness of the approximate probability $1-\alpha$ to the exact probability depends on both the underlying distribution and the sample size. When the underlying distribution is unimodal (has only one mode) and continuous, the approximation is usually quite good for even small **n**, such as $n=5$. As the underlying distribution becomes less normal (**i.e.**, badly skewed or discrete), a larger sample size might be required to keep reasonably accurate approximation. But, in all cases, an **n** of at least 30 is usually quite adequate.

NOTE: Confidence Intervals II