BEST CRITICAL REGIONS*

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Abstract

This course is a short series of lectures on Introductory Statistics. Topics covered are listed in the Table of Contents. The notes were prepared by Ewa Paszek and Marek Kimmel. The development of this course has been supported by NSF 0203396 grant.

1 BEST CRITICAL REGIONS

In this paragraph, let consider the properties a satisfactory test should posses.

1.1

Definition 1:
1. Consider the test of the sample null hypothesis $H_0 : \theta = \theta_0$ against the simple alternative hypothesis $H_1 : \theta = \theta_1$.
2. Let **C** be a critical region of size α; that is, $\alpha = P(C; \theta_0)$. Then **C** is **a best critical region of size** α if, for every other critical region **D** of size $\alpha = P(D; \theta_0)$, we have that

$$P(C; \theta_1) \geq P(D; \theta_1).$$

That is, when $H_1 : \theta = \theta_1$ is true, the probability of rejecting $H_0 : \theta = \theta_0$ using the critical region **C** is at least as great as the corresponding probability using any other critical region **D** of size α.

Thus a best critical region of size α is the critical region that has the greatest power among all critical regions for a best critical region of size α. **The Neyman-Pearson lemma** gives sufficient conditions for a best critical region of size α.

1.2

Theorem 1: Neyman-Pearson Lemma
Let $X_1, X_2, ..., X_n$ be a random sample of size **n** from a distribution with p.d.f. $f(x; \theta)$, where θ_0 and θ_1 are two possible values of θ.

Denote the joint p.d.f. of $X_1, X_2, ..., X_n$ by the likelihood function

$$L(\theta) = L(\theta; x_1, x_2, ..., x_n) = f(x_1; \theta) f(x_2; \theta) \cdots f(x_{n.}; \theta).$$

If there exist a positive constant **k** and a subset **C** of the sample space such that

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1. $P\left[(X_1, X_2, ..., X_n) \in C; \theta_0\right] = \alpha$,
2. $\frac{L(\theta_0)}{L(\theta_1)} \leq k$ for $(x_1, x_2, ..., x_n) \in C$,
3. $\frac{L(\theta_0)}{L(\theta_1)} \geq k$ for $(x_1, x_2, ..., x_n) \in C$'.

Then **C** is a best critical region of size α for testing the simple null hypothesis $H_0 : \theta = \theta_0$ against the simple alternative hypothesis $H_1 : \theta = \theta_1$.

1.3

For a realistic application of the Neyman-Pearson lemma, consider the following, in which the test is based on a random sample from a normal distribution.

Example 1

Let $X_1, X_2, ..., X_n$ be a random sample from a normal distribution $N(\mu, 36)$. We shall find the best critical region for testing the simple hypothesis $H_0 : \mu = 50$ against the simple alternative hypothesis $H_1 : \mu = 55$. Using the ratio of the likelihood functions, namely $L(50)/L(55)$, we shall find those points in the sample space for which this ratio is less than or equal to some constant **k**.

That is, we shall solve the following inequality:

$$\begin{aligned}\frac{L(50)}{L(55)} &= \frac{(72\pi)^{-n/2}\exp\left[-\left(\frac{1}{72}\right)\sum_1^n (x_i-50)^2\right]}{(72\pi)^{-n/2}\exp\left[-\left(\frac{1}{72}\right)\sum_1^n (x_i-55)^2\right]} \\ &= \exp\left[-\left(\frac{1}{72}\right)\left(10\sum_1^n x_i + n50^2 - n55^2\right)\right] \leq k.\end{aligned}$$

If we take the natural logarithm of each member of the inequality, we find that

$$-10\sum_1^n x_i - n50^2 + n55^2 \leq (72)\ln k.$$

Thus,

$$\frac{1}{n}\sum_1^n x_i \geq -\frac{1}{10n}\left[n50^2 - n55^2 + (72)\ln k\right]$$

Or equivalently, $\overline{x} \geq c$, where $c = -\frac{1}{10n}\left[n50^2 - n55^2 + (72)\ln k\right]$.

Thus $L(50)/L(55) \leq k$ is equivalent to $\overline{x} \geq c$.

A best critical region is, according to the Neyman-Pearson lemma,

$$C = \{(x_1, x_2, ..., x_n) : \overline{x} \geq c\},$$

where **c** is selected so that the size of the critical region is α. Say **n**=16 and **c**=53. Since $\overline{X}$ is $N(50, 36/16)$ under H_0 we have

$$\alpha = P\left(\overline{X} \geq 53; \mu = 50\right) = P\left[\frac{\overline{X} - 50}{6/4} \geq \frac{3}{6/4}; \mu = 50\right] = 1 - \Phi(2) = 0.0228.$$

The example 1 illustrates what is often true, namely, that the inequality

$$\frac{L(\theta_0)}{L(\theta_1)} \leq k$$

can be expressed in terms of a function $u(x_1, x_2, ..., x_n)$ say,

$$u(x_1, x_2, ..., x_n) \leq c_1$$

or

$$u\left(x_1, x_2, ..., x_n\right) \geq c_2,$$

where c_1 and c_2 is selected so that the size of the critical region is α . Thus the test can be based on the statistic $u\left(X_1, ..., X_n\right)$. Also, for illustration, if we want α to be a given value, say 0.05, we would then choose our c_1 and c_2. In example1, with α=0.05, we want

$$0.05 = P\left(\overline{X} \geq c; \mu = 50\right) = P\left(\frac{\overline{X} - 50}{6/4} \geq \frac{c - 50}{6/4}; \mu = 50\right) = 1 - \Phi\left(\frac{c - 50}{6/4}\right).$$

Hence it must be true that $(c - 50) / (3/2) = 1.645$, or equivalently, $c = 50 + \frac{3}{2}(1.645) \approx 52.47$.

1.4

Example 2

Let$X_1, X_2, ..., X_n$ denote a random sample of size **n** from a Poisson distribution with mean λ. A best critical region for testing $H_0 : \lambda = 2$ against $H_1 : \lambda = 5$ is given by

$$\frac{L(2)}{L(5)} = \frac{2^{\sum x_i} e^{-2n}}{x_1! x_2! \cdots x_n!} \frac{x_1! x_2! \cdots x_n!}{5^{\sum x_i} e^{-5n}} \leq k.$$

The inequality is equivalent to $\left(\frac{2}{5}\right)^{\sum x_i} e^{3n} \leq k$ and $\left(\sum x_i\right) \ln\left(\frac{2}{5}\right) + 3n \leq \ln k$.
Since $\ln(2/5) < 0$, this is the same as

$$\sum_{i=1}^{n} x_i \geq \frac{\ln k - 3n}{\ln(2/5)} = c.$$

If **n**=4 and **c**=13, then

$$\alpha = P\left(\sum_{i=1}^{4} X_i \geq 13; \lambda = 2\right) = 1 - 0.936 = 0.064,$$

from the tables, since $\sum_{i=1}^{4} X_i$ has a Poisson distribution with mean 8 when λ=2.

When $H_0 : \theta = \theta_0$ and $H_1 : \theta = \theta_1$ are both simple hypotheses, a critical region of size α is a best critical region if the probability of rejecting H_0 when H_1 is true is a maximum when compared with all other critical regions of size α. The test using the best critical region is called **a most powerful test** because it has the greatest value of the power function at $\theta = \theta_1$ when compared with that of other tests of significance level α. If H_1 is a composite hypothesis, the power of a test depends on each simple alternative in H_1 .

1.4.1

Definition 2:

A test, defined by a critical region **C** of size α, is **a uniformly most powerful test** if it is a most powerful test against each simple alternative in H_1. The critical region **C** is called **a uniformly most powerful critical region of size** α.

Let now consider the example when the alternative is composite.

Example 3

Let $X_1, X_2, ..., X_n$ be a random sample from $N(\mu, 36)$. We have seen that when testing $H_0 : \mu = 50$ against $H_1 : \mu = 55$, a best critical region **C** is defined by

$$C = \{(x_1, x_2, ..., x_n) : \overline{x} \geq c\},$$

where **c** is selected so that the significance level is α. Now consider testing $H_0 : \mu = 50$ against the one-sided composite alternative hypothesis $H_1 : \mu > 50$. For each simple hypothesis in H_1, say $\mu = \mu_1$ the quotient of the likelihood functions is

$$\begin{aligned}\frac{L(50)}{L(\mu_1)} &= \frac{(72\pi)^{-n/2}\exp\left[-\left(\frac{1}{72}\right)\sum_1^n (x_i-50)^2\right]}{(72\pi)^{-n/2}\exp\left[-\left(\frac{1}{72}\right)\sum_1^n (x_i-\mu_1)^2\right]} \\ &= \exp\left[-\left(\frac{1}{72}\right)\left\{2\left(\mu_1-50\right)\sum_1^n x_i + n\left(50^2-\mu_1^2\right)\right\}\right].\end{aligned}$$

Now $L(50)/L(\mu_1) \leq k$ if and only if

$$\overline{x} \geq \frac{(-72)\ln(k)}{2n(\mu_1-50)} + \frac{50+\mu_1}{2} = c.$$

Thus the best critical region of size α for testing $H_0 : \mu = 50$ against $H_1 : \mu = \mu_1$, where $\mu_1 > 50$, is given by

$$C = \{(x_1, x_2, ..., x_n) : \overline{x} \geq c\},$$

where is selected such that

$$P\left(\overline{X} \geq c; H_0 : \mu = 50\right) = \alpha.$$

2

NOTE: the same value of **c** can be used for each $\mu_1 > 50$, but of course **k** does not remain the same. Since the critical region **C** defines a test that is most powerful against each simple alternative $\mu_1 > 50$, this is a uniformly most powerful test, and **C** is a uniformly most powerful critical region if size α. Again if α=0.05, then $c \approx 52.47$.