Features of projectile motion (application)*

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Abstract

Solving problems is an essential part of the understanding process.

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

1 Hints on solving problems

1: In general, we should rely on analysis in two individual directions as linear motion.

2: Wherever possible, we should use the formula directly as available for time of flight, maximum height and horizontal range.

3: We should be aware that time of flight and maximum height are two attributes of projectile motion, which are obtained by analyzing motion in vertical direction. For determining time of flight, the vertical displacement is zero; whereas for determining maximum height, vertical component of velocity is zero.

4: However, if problem has information about motion in horizontal direction, then it is always advantageous to analyze motion in horizontal direction. It is so because motion in horizontal direction is uniform motion and analysis in this direction is simpler.

5: The situation, involving quadratic equations, may have three possibilities : (i) quadratic in time "t" (ii) quadratic in displacement or position "x" and (iii) quadratic in "tanθ" i.e."θ". We should use appropriate equations in each case as discussed in the module titled " Features of projectile motion [1]".

2 Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the features of projectile motion. The questions are categorized in terms of the characterizing features of the subject matter :

- Time of flight
- Horizontal range
- Maximum height

*Version 1.9: Oct 15, 2008 12:29 pm -0500

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[1] "Features of projectile motion" <http://cnx.org/content/m13847/latest/>

- Height attained by a projectile
- Composition of motion
- Projectile motion with wind/drag force

3 Time of flight

Example 1

Problem : The speed of a particle, projected at 60 °, is 20 m/s at the time of projection. Find the time interval for projectile to loose half its initial speed.

Solution : Here, we see that final and initial speeds (not velocity) are subject to given condition. We need to use the given condition with appropriate expressions of speeds for two instants.

$$u = \sqrt{\left(u_x^2 + u_y^2\right)}$$

$$v = \sqrt{\left(v_x^2 + v_y^2\right)}$$

According to question,

$$u = 2v$$

$$\Rightarrow u^2 = 4v^2$$

$$\Rightarrow u_x^2 + u_y^2 = 4\left(v_x^2 + v_y^2\right)$$

But, horizontal component of velocity remains same. Hence,

$$\Rightarrow u_x^2 + u_y^2 = 4\left(u_x^2 + v_y^2\right)$$

Rearranging for vertical velocity :

$$v_y^2 = u_y^2 - 3u_x^2 = \left(20\text{sin}60^0\right)^2 - 3X\left(20\text{cos}60^0\right)^2$$

$$v_y^2 = 20X\frac{3}{4} - 3X20X\frac{1}{4} = 0$$

The component of velocity in vertical direction becomes zero for the given condition. This means that the projectile has actually reached the maximum height for the given condition. The time to reach maximum height is half of the time of flight :

$$t = \frac{u\text{sin}\theta}{g} = \frac{20X\text{sin}60^0}{10} = \sqrt{3} \quad s$$

4 Horizontal range

Example 2

Problem : A man can throw a ball to a greatest height denoted by "h". Find the greatest horizontal distance that he can throw the ball (consider g = 10 $m\ /\ s^2$).

Solution : The first part of the question provides the information about the initial speed. We know that projectile achieves greatest height in vertical throw. Let "u" be the initial speed. We can, now, apply equation of motion " $v^2 = u^2 + 2gy$ " for vertical throw. We use this form of equation as we want to relate initial speed with the greatest height.

Here, v = 0; a = -g

$$\Rightarrow\ 0 = u^2 - 2gh$$
$$u^2 = 2gh$$

The projectile, on the other hand, attains greatest horizontal distance for the angle of projection, $\theta = 45\,^\circ$. Accordingly, the greatest horizontal distance is :

$$R_{\max} = \frac{u^2 \sin 2\ X\ 45\,^\circ}{g} = \frac{u^2 \sin 90\,^\circ}{g} = \frac{u^2}{g} = \frac{2gh}{g} = 2h$$

Example 3

Problem : A bullet from a gun is fired at a muzzle speed of 50 m/s to hit a target 125 m away at the same horizontal level. At what angle from horizontal should the gun be aimed to hit the target (consider g = 10 $m\ /\ s^2$) ?

Solution : Here, horizontal range is given. We can find out the angle of projection from horizontal direction, using expression of horizontal range :

$$R = \frac{u^2 \sin 2\theta}{g}$$
$$\Rightarrow\ \sin 2\theta = \frac{gR}{u^2} = \frac{10\ x\ 125}{50^2} = \frac{1}{2}$$
$$\Rightarrow\ \sin 2\theta = \sin 30^0$$
$$\Rightarrow\ \theta = 15^0$$

Example 4

Problem : A projectile has same horizontal range for a given projection speed for the angles of projections θ_1 and θ_2 ($\theta_2 > \theta_1$) with the horizontal. Find the ratio of the times of flight for the two projections.

Solution : The ratio of time of flight is :

$$\frac{T_1}{T_2} = \frac{u \sin\theta_1}{u \sin\theta_2} = \frac{\sin\theta_1}{\sin\theta_2}$$

For same horizontal range, we know that :

$$\theta_2 = \left(90^0 - \theta_1 \right)$$

Putting this, we have :

$$\frac{T_1}{T_2} = \frac{\sin\theta_1}{\sin\left(90^0 - \theta_1 \right)} = \tan\theta_1$$

Example 5

Problem : A projectile, thrown at an angle 15 ° with the horizontal, covers a horizontal distance of 1000 m. Find the maximum distance the projectile can cover with the same speed (consider g = 10 $m\ /\ s^2$).

Solution : The range of the projectile is given by :

$$R = \frac{u^2 \sin 2\theta}{g}$$

Here, R = 1000 m, g = 10 m / s^2 , $\theta = 15°$

$$\Rightarrow 1000 = \frac{u^2 \sin\left(2 \ x \ 15^0\right)}{g} = \frac{u^2}{2g}$$
$$\Rightarrow \frac{u^2}{2g} = 1000\, m$$

Now, the maximum range (for $\theta = 45°$) is :

$$\Rightarrow R_{\max} = \frac{u^2}{g} = 2000\, m$$

5 Maximum height

Example 6

Problem : Two balls are projected from the same point in the direction inclined at 60° and 30° respectively with the horizontal. If they attain the same height, then the ratio of speeds of projection is :

Solution : Since the projectiles attain same height,

Projectile motion

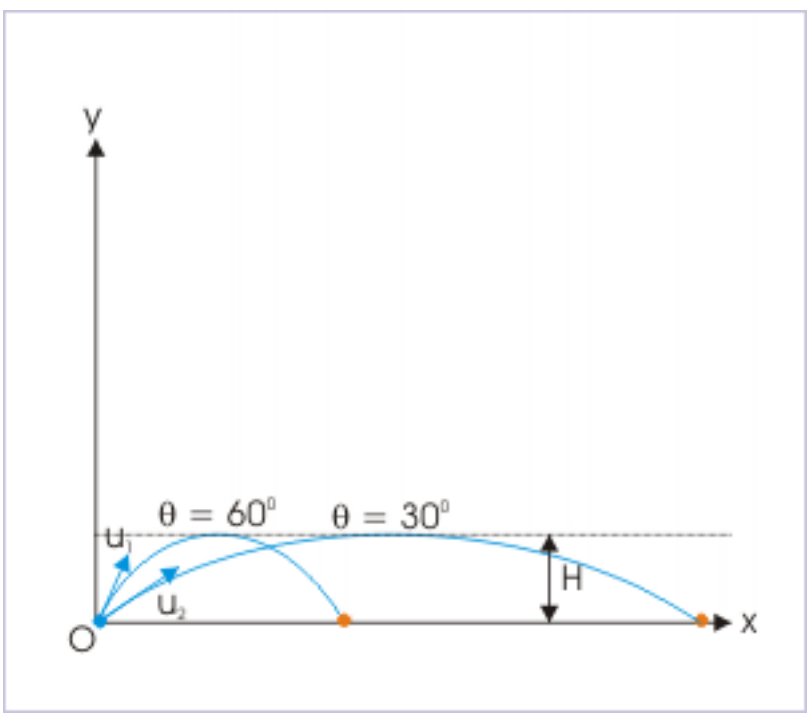

Figure 1

$$
\begin{aligned}
H_1 &= H_2 = H \\
\frac{{u_1}^2 \sin^2 60^0}{2g} &= \frac{{u_2}^2 \sin^2 30^0}{2g} \\
\frac{{u_1}^2 \ X \ 3}{4} &= \frac{{u_2}^2 \ X \ 1}{4} \\
{u_1}^2 &: {u_2}^2 = 1 : 3 \\
\Rightarrow u_1 &: u_2 = 1 : \sqrt{3}
\end{aligned}
$$

Example 7

Problem : A projectile is thrown vertically up, whereas another projectile is thrown at an angle θ with the vertical. Both of the projectiles stay in the air for the same time (neglect air resistance). Find the ratio of maximum heights attained by two projectiles.

Solution : Let u_1 and u_2 be the speeds of projectiles for vertical and non-vertical projections. The times of the flight for vertical projectile is given by :

$$T_1 = \frac{2u_1}{g}$$

We note here that the angle is given with respect to vertical - not with respect to horizontal as the usual case. As such, the expression of time of flight consists of cosine term :

$$T_2 = \frac{2u_2 \cos\theta}{g}$$

As, time of flight is same,

$$
\begin{aligned}
\frac{2u_1}{g} &= \frac{2u_2 \cos\theta}{g} \\
u_1 &= u_2 \cos\theta
\end{aligned}
$$

On the other hand, the maximum heights attained in the two cases are :

$$
\begin{aligned}
H_1 &= \frac{{u_1}^2}{2g} \\
H_2 &= \frac{{u_2}^2 \cos^2\theta}{2g}
\end{aligned}
$$

Using the relation $u_1 = u_2 \cos\theta$ as obtained earlier, we have :

$$
\begin{aligned}
H_2 &= \frac{{u_1}^2}{2g} \\
\Rightarrow H_1 &= H_2
\end{aligned}
$$

NOTE: The result is intuitive about the nature of projectile. The time of flight and vertical height both are consideration of motion in vertical direction. Since times of flight in both cases are same, the vertical components of two projectiles should be same. Otherwise, times of flight will be different. Now, if vertical component are same, then maximum heights have to be same.

6 Height attained by a projectile

Example 8

Problem : The times for attaining a particular vertical elevation during projectile motion are t_1 and t_2 . Find time of flight, T, in terms of t_1 and t_2 .

Solution : We can answer this question analytically without using formula. Let the positions of the projectile at two time instants be "A" and "B", as shown in the figure. The time periods t_1 and t_2 denotes time taken by the projectile to reach points "A" and "B" respectively. Clearly, time of flight, T, is equal to time taken to travel the curve OAB (t_2) plus the time taken to travel the curve BC.

Projectile motion

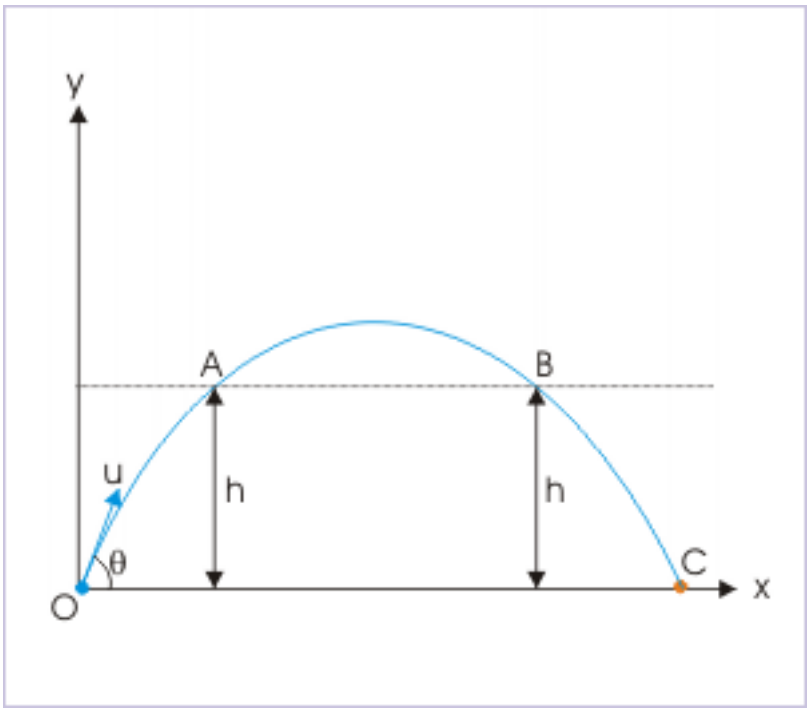

Figure 2

Now, projectile takes as much time to travel the curve OA, as it takes to travel curve BC. This is so, because the time of travel of equal vertical displacement in either direction (up or down) in vertical motion under gravity is same. Since the time of travel for curve OA is t_1 , the time of travel for curve BC is also t_1 . Thus, the total time of flight is :

$$T = t_1 + t_2$$

Alternatively,
As the heights attained are equal,

$$
\begin{aligned}
& h_1 = h_2 \\
& \Rightarrow u_y t_1 - \tfrac{1}{2} g {t_1}^2 = u_y t_2 - \tfrac{1}{2} g {t_2}^2 \\
& \Rightarrow u_y \left(t_2 - t_1 \right) = \tfrac{1}{2} g \left({t_2}^2 - {t_1}^2 \right) = \tfrac{1}{2} g \left(t_2 + t_1 \right) \left(t_2 - t_1 \right) \\
& \Rightarrow t_2 + t_1 = \tfrac{2 u_y}{g} = T
\end{aligned}
$$

7 Composition of motion

Example 9

Problem : The position of a projectile projected from the ground is :

$$
\begin{aligned}
x &= 3t \\
y &= \left(4t - 2t^2 \right)
\end{aligned}
$$

where "x" and "y" are in meters and "t" in seconds. The position of the projectile is (0,0) at the time of projection. Find the speed with which the projectile hits the ground.

Solution : When the projectile hits the ground, y = 0,

$$
\begin{aligned}
& 0 = \left(4t - 2t^2 \right) \\
& \Rightarrow 2t^2 - 4t = t \left(2t - 2 \right) = 0 \\
& \Rightarrow t = 0 , \quad t = 2\, s
\end{aligned}
$$

Here t = 0 corresponds to initial condition. Thus, projectile hits the ground in 2 s. Now velocities in two directions are obtained by differentiating given functions of the coordinates,

$$
\begin{aligned}
v_x &= \tfrac{x}{t} = 3 \\
v_y &= \tfrac{y}{t} = 4 - 4t
\end{aligned}
$$

Now, the velocities for t = 2 s,

$$
\begin{aligned}
v_x &= \tfrac{x}{t} = 3\, m / s \\
v_y &= \tfrac{y}{t} = 4 - 4\, x\ 2 = -4\, m / s
\end{aligned}
$$

The resultant velocity of the projectile,

$$
v = \sqrt{\left({v_x}^2 + {v_y}^2 \right)} = \sqrt{\left\{ 3^2 + \left(-4 \right)^2 \right\}} = 5\, m / s
$$

Example 10

Problem : A projectile, thrown from the foot of a triangle, lands at the edge of its base on the other side of the triangle. The projectile just grazes the vertex as shown in the figure. Prove that :

Projectile motion

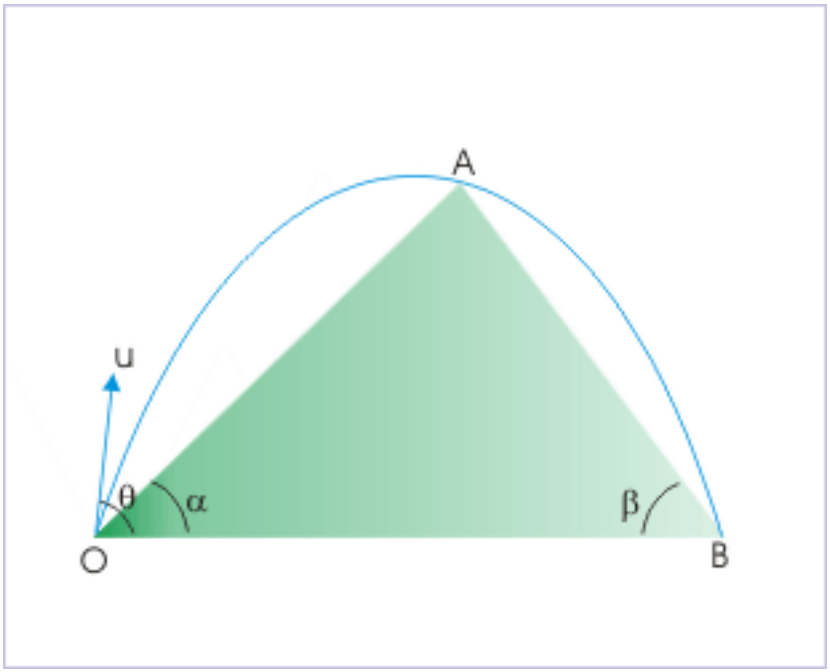

Figure 3: The projectile grazes the vertex of the triangle.

$$\tan\alpha + \tan\beta = \tan\theta$$

where "θ" is the angle of projection as measured from the horizontal.

Solution : In order to expand trigonometric ratio on the left side, we drop a perpendicular from the vertex of the triangle "A" to the base line OB to meet at a point C. Let x,y be the coordinate of vertex "A", then,

Projectile motion

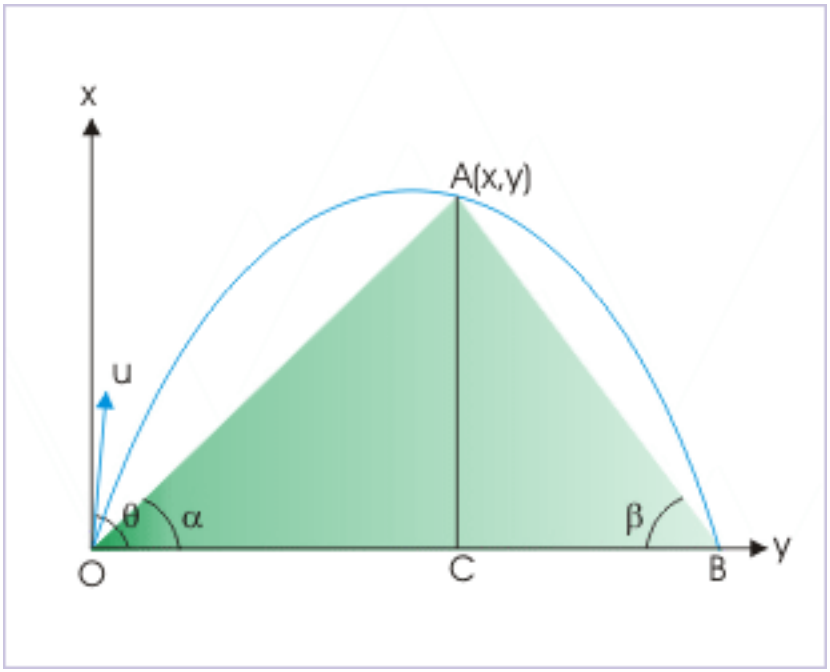

Figure 4: The projectile grazes the vertex of the triangle.

$$\tan\alpha = \frac{AC}{OC} = \frac{y}{x}$$

and

$$\tan\beta = \frac{AC}{BC} = \frac{y}{(R-x)}$$

Thus,

$$\tan\alpha + \tan\beta = \frac{y}{x} + \frac{y}{(R-x)} = \frac{yR}{x\,(R-x)}$$

Intuitively, we know the expression is similar to the expression involved in the equation of projectile motion that contains range of projectile,

$$\Rightarrow y = x\tan\theta\left(1 - \frac{x}{R}\right)$$

$$\Rightarrow \tan\theta = \frac{yR}{x\,(R-x)}$$

Comparing equations,

$$\Rightarrow \tan\alpha + \tan\beta = \tan\theta$$

8 Projectile motion with wind/drag force

Example 11

Problem : A projectile is projected at angle "θ" from the horizontal at the speed "u". If an acceleration of "g/2" is applied to the projectile due to wind in horizontal direction, then find the new time of flight, maximum height and horizontal range.

Solution : The acceleration due to wind affects only the motion in horizontal direction. It would, therefore, not affect attributes like time of flight or maximum height that results exclusively from the consideration of motion in vertical direction. The generic expressions of time of flight, maximum height and horizontal range of flight with acceleration are given as under :

$$T = \frac{2u_y}{g}$$

$$H = \frac{u_y^2}{2g} = \frac{gT^2}{4}$$

$$R = \frac{u_x u_y}{g}$$

The expressions above revalidate the assumption made in the beginning. We can see that it is only the horizontal range that depends on the component of motion in horizontal direction. Now, considering accelerated motion in horizontal direction, we have :

$$x = R\prime = u_x T + \frac{1}{2} a_x T^2$$

$$\Rightarrow R\prime = u_x T + \frac{1}{2}\left(\frac{g}{2}\right) T^2$$

$$R\prime = R + H$$

Example 12

Problem : A projectile is projected at angle "θ" from the horizontal at the speed "u". If an acceleration of g/2 is applied to the projectile in horizontal direction and a deceleration of g/2 in vertical direction, then find the new time of flight, maximum height and horizontal range.

Solution : The acceleration due to wind affects only the motion in horizontal direction. It would, therefore, not affect attributes resulting exclusively from the consideration in vertical direction. It is only the horizontal range that will be affected due to acceleration in horizontal direction. On the other hand, deceleration in vertical direction will affect all three attributes.

1: Time of flight

Let us work out the effect on each of the attribute. Considering motion in vertical direction, we have :

$$y = u_y T + \frac{1}{2} a_y T^2$$

For the complete flight, y = 0 and t = T. Also,

$$a_y = -\left(g + \frac{g}{2}\right) = -\frac{3g}{2}$$

Putting in the equation,

$$\Rightarrow 0 = u_y T - \frac{1}{2} X \frac{3g}{2} X T^2$$

Neglecting T = 0,

$$\Rightarrow T = \frac{4u_y}{3g} = \frac{4u\sin\theta}{3g}$$

2: Maximum height
For maximum height, $v_y = 0$,

$$0 = u_y^2 - 2X\frac{3g}{2}XH$$

$$H = \frac{u_y^2}{3g} = \frac{u^2\sin^2\theta}{3g}$$

2: Horizontal range
Now, considering accelerated motion in horizontal direction, we have :

$$x = R = u_xT + \frac{1}{2}a_xT^2$$

$$R = u_x\left(\frac{4u_y}{g}\right) + \frac{1}{2}\left(\frac{g}{2}\right)\left(\frac{4u_y}{g}\right)^2$$

$$\Rightarrow R = \left(\frac{4u_y}{g}\right)\left[u_x + \frac{1}{2}\left(\frac{g}{2}\right)\left(\frac{4u_y}{g}\right)\right]$$

$$\Rightarrow R = \left(\frac{4u_y}{g}\right)\{u_x + u_y\}$$

$$\Rightarrow R = \left(\frac{4u^2\sin\theta}{g}\right)[\cos\theta + \sin\theta]$$