m19 - Wavlet-Based Signal Analysis*

C. Sidney Burrus

This work is produced by OpenStax-CNX and licensed under the
Creative Commons Attribution License 2.0†

Abstract

A multiresolution formulation of signal decomposition give a signal expansion in terms of basis functions called wavelets.

1 Wavelet-Based Signal Analysis

There are wavelet systems and transforms analogous to the DFT, Fourier series, discrete-time Fourier transform, and the Fourier integral. We will start with the discrete wavelet transform (DWT) which is analogous to the Fourier series and probably should be called the wavelet series [2]. Wavelet analysis can be a form of time-frequency analysis which locates energy or events in time and frequency (or scale) simultaneously. It is somewhat similar to what is called a short-time Fourier transform or a Gabor transform or a windowed Fourier transform.

The history of wavelets and wavelet based signal processing is fairly recent. Its roots in signal expansion go back to early geophysical and image processing methods and in DSP to filter bank theory and subband coding. The current high interest probably started in the late 1980's with the work of Mallat, Daubechies, and others. Since then, the amount of research, publication, and application has exploded. Two excellent descriptions of the history of wavelet research and development are by Hubbard [4] and by Daubechies [?] and a projection into the future by Sweldens [?] and Burrus [1].

1.1 The Basic Wavelet Theory

The ideas and foundations of the basic dyadic, multiresolution wavelet systems are now pretty well developed, understood, and available [2][3][8][7]. The first basic requirement is that a set of expansion functions (usually a basis) are generated from a single "mother" function by translation and scaling. For the discrete wavelet expansion system, this is

$$\phi_{j,k}(t) = \phi\left(2^j t - k\right) \tag{1}$$

where j, k are integer indices for the series expansion of the form

$$f(t) = \sum_{j,k} c_{j,k} \phi_{j,k}(t). \tag{2}$$

*Version 1.1: Sep 17, 2006 12:46 pm -0500
†http://creativecommons.org/licenses/by/2.0/

The coefficients $c_{j,k}$ are called the discrete wavelet transform of the signal $f(t)$. This use of translation and scale to create an expansion system is the foundation of all so-called first generation wavelets [?].

The system is somewhat similar to the Fourier series described in () with frequencies being related by powers of two rather than an integer multiple and the translation by k giving only the two results of cosine and sine for the Fourier series.

The second almost universal requirement is that the wavelet system generates a multiresolution analysis (MRA). This means that a low resolution function (low scale j) can be expanded in terms of the same function at a higher resolution (higher j). This is stated by requiring that the generator of a MRA wavelet system, called a scaling function $\phi(t)$, satisfies

$$\phi(t) = \sum_n h(n)\,\phi(2t-n)\,. \tag{3}$$

This equation, called the refinement equation or the MRA equation or basic recursion equation, is similar to a differential equation in that its solution is what defines the basic scaling function and wavelet [?][2].

The current state of the art is that most of the necessary and sufficient conditions on the coefficients $h(n)$ are known for the existence, uniqueness, orthogonality, and other properties of $\phi(t)$. Some of the theory parallels Fourier theory and some does not.

A third important feature of a MRA wavelet system is a discrete wavelet transform (DWT) can be calculated by a digital filter bank using what is now called Mallat's algorithm. Indeed, this connection with digital signal processing (DSP) has been a rich source of ideas and methods. With this filter bank, one can calculate the DWT of a length-N digital signal with order N operations. This means the number of multiplications and additions grows only linearly with the length of the signal. This compares with $N\log(N)$ for an FFT and N^2 for most methods and worse than that for some others.

These basic ideas came from the work of Meyer, Daubechies, Mallat, and others but for a time looked like a solution looking for a problem. Then a second phase of research showed there are many problems to which the wavelet is an excellent solution. In particular, the results of Donoho, Johnstone, Coifman, Beylkin, and others opened another set of doors.

1.2 Generalization of the Basic Wavelet System

After (in some cases during) much of the development of the above basic ideas, a number of generalizations [2] were made. They are listed below:

1. A larger integer scale factor than $M = 2$ can be used to give a more general M-band refinement equation [6]
$$\phi(t) = \sum_n h(n)\,\phi(Mt-n) \tag{4}$$
than the "dyadic" or octave based equation ((3)). This also gives more than two channels in the accompanying filter bank. It allows a uniform frequency resolution rather than the resulting logarithmic one for $M = 2$.
2. The wavelet system called a wavelet packet is generated by "iterating" the wavelet branches of the filter bank to give a finer resolution to the wavelet decomposition. This was suggested by Coifman and it too allows a mixture of uniform and logarithmic frequency resolution. It also allows a relatively simple adaptive system to be developed which has an automatically adjustable frequency resolution based on the properties of the signal.
3. The usual requirement of translation orthogonality of the scaling function and wavelets can be relaxed to give what is called a biorthogonal system [?]. If the expansion basis is not orthogonal, a dual basis can be created that will allow the usual expansion and coefficient calculations to be made. The main disadvantage is the loss of a Parseval's theorem which maintains energy partitioning. Nevertheless, the greater flexibility of the biorthogonal system allows superior performance in many compression and denoising applications.

4. The basic refinement equation ((3)) gives the scaling function in terms of a compressed version of itself (self-similar). If we allow two (or more) scaling functions, each being a weighted sum of a compress version of both, a more general set of basis functions results. This can be viewed as a vector of scaling functions with the coefficients being a matrix now. Once again, this generalization allows more flexibility in the characteristics of the individual scaling functions and their related multi-wavelets. These are called multi-wavelet systems and are still being developed.
5. One of the very few disadvantages of the discrete wavelet transform is the fact it is not shift invariant. In other words, if you shift a signal in time, its wavelet transform not only shifts, it changes character! For many applications in denoising and compression, this is not desirable although it may be tolerable. The DWT can be made shift-invariant by calculating the DWT of a signal for all possible shifts and adding (or averaging) the results. That turns out to be equivalent to removing all of the down-samplers in the associated filter bank (an undecimated filter bank), which is also equivalent to building an overdetermined or redundant DWT from a traditional wavelet basis. This overcomplete system is similar to a "tight frame" and maintains most of the features of an orthogonal basis yet is shift invariant. It does, however, require $N\log(N)$ operations.
6. Wavelet systems are easily modified to being an adaptive system where the basis adjusts itself to the properties of the signal or the signal class. This is often done by starting with a large collection or library of expansion systems and bases. A subset is adaptively selected based on the efficiency of the representation using a process sometimes called pursuit. In other words, a set is chosen that will result in the smallest number of significant expansion coefficients. Clearly, this is signal dependent, which is both its strength and its limitation. It is nonlinear.
7. One of the most powerful structures yet suggested for using wavelets for signal processing is to first take the DWT, then do a point-wise linear or nonlinear processing of the DWT, finally followed by an inverse DWT. Simply setting some of the wavelet domain expansion terms to zero results in linear wavelet domain filtering, similar to what would happen if the same were done with Fourier transforms. Donoho [?][?] and others have shown by using some form of nonlinear thresholding of the DWT, one can achieve near optimal denoising or compression of a signal. The concentrating or localizing character of the DWT allows this nonlinear thresholding to be very effective.

The present state of activity in wavelet research and application shows great promise based on the above generalizations and extensions of the basic theory and structure [1]. We now have conferences, workshops, articles, newsletters, books, and email groups that are moving the state of the art forward. More details, examples, and software are given in [2][?][5].

References

[1] C. Sidney Burrus. Wavelet based signal processing: Where are we and where are we going?, plenary talk. In *Proceedings of the International Conference on Digital Signal Processing*, volume I, pages 3–5, Santorini, Greece, July 1997.

[2] Ramesh A. Gopinath C. Sidney Burrus and Haitao Guo. *Introduction to Wavelets and the Wavelet Transform.* Prentice Hall, Upper Saddle River, NJ, 1998.

[3] Ingrid Daubechies. *Ten Lectures on Wavelets.* SIAM, Philadelphia, PA, 1992. Notes from the 1990 CBMS-NSF Conference on Wavelets and Applications at Lowell, MA.

[4] Barbara Burke Hubbard. *The World According to Wavelets.* A K Peters, Wellesley, MA, 1996. Second Edition 1998.

[5] Georges Oppenheim Michel Misiti, Yves Misiti and Jean-Michel Poggi. *Wavelet Toolbox User's Guide.* The MathWorks, Inc., Natick, MA, 1996.

[6] R. A. Gopinath P. Steffen, P. N. Heller and C. S. Burrus. Theory of regular m-band wavelet bases. *IEEE Transactions on Signal Processing*, 41(12):3497–3511, December 1993. Special Transaction issue on wavelets; Rice contribution also in Tech. Report No. CML TR-91-22, Nov. 1991.

[7] Gilbert Strang and T. Nguyen. *Wavelets and Filter Banks*. Wellesley-Cambridge Press, Wellesley, MA, 1996.

[8] Martin Vetterli and Jelena Kova\vcevi263;. *Wavelets and Subband Coding*. Prentice-Hall, Upper Saddle River, NJ, 1995.