m25 - Fourier Techniques for Sampling*

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Abstract

Aliasing caused by sampling can best be developed with Fourier transforms.

1 Fourier Techniques

We first start with the most basic sampling ideas based on various forms of Fourier transforms [2][1][3].

The Spectrum of a Continuous-Time Signal and the Fourier Transform

Although in many cases digital signal processing views the signal as simple sequence of numbers, here we are going to pose the problem as originating with a function of continuous time. The fundamental tool is the classical Fourier transform defined by

$$F(\omega) = \int f(t) e^{-j\omega t} dt \tag{1}$$

and its inverse

$$f(t) = \frac{1}{2\pi} \int F(\omega) e^{j\omega t} d\omega. \tag{2}$$

where $j = \sqrt{-1}$. The Fourier transform of a signal is called its spectrum and it is complex valued with a magnitude and phase.

If the signal is periodic with period $f(t) = f(t+P)$, the Fourier transform does not exist as a function (it may as a distribution) therefore the spectrum is defined as the set of Fourier series coefficients

$$C(k) = \frac{1}{P} \int_0^P f(t) e^{-j2\pi kt/P} dt \tag{3}$$

with the expansion having the form

$$f(t) = \sum_k C(k) e^{j2\pi kt/P}. \tag{4}$$

The functions $g_k(t) = e^{j2\pi kt/P}$ form an orthogonal basis for periodic functions and ((3)) is the inner product $C(k) =< f(t), g_k(t) >$.

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For the non-periodic case in ((1)) the spectrum is a function of continuous frequency and for the periodic case in ((3)), the spectrum is a number sequence (a function of discrete frequency). The Spectrum of a Sampled Signal and the DTFT

The discrete-time Fourier transform (DTFT) as defined in terms samples of a continuous function is

$$F_d(\omega) = \sum_n f(Tn) e^{-j\omega Tn} \tag{5}$$

and its inverse

$$f(Tn) = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} F_d(\omega) e^{j\omega Tn} d\omega \tag{6}$$

can be derived by noting that $F_d(\omega)$ is periodic with period $P = 2\pi/T$ and, therefore, it can be expanded in a Fourier series with ((6)) resulting from calculating the series coefficients using ((3)).

The spectrum of a discrete-time signal is defined as the DTFT of the samples of a continuous-time signal given in ((5)). Samples of the signal are given by the inverse DTFT in ((6)) but they can also be obtained by directly sampling $f(t)$ in ((2)) giving

$$f(Tn) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega Tn} d\omega \tag{7}$$

which can be rewritten as an infinite sum of finite integrals in the form

$$\begin{aligned} f(Tn) &= \frac{1}{2\pi} \sum_\ell \int_0^{2\pi/T} F(\omega + 2\pi\ell/T) e^{j(\omega+2\pi\ell/T)Tn} d\omega \\ &= \frac{1}{2\pi} \int_0^{2\pi/T} \left[\sum_\ell F(\omega + 2\pi\ell/T) \right] e^{j(\omega+2\pi\ell/T)Tn} d\omega \end{aligned} \tag{8}$$

where $F_p(\omega)$ is a periodic function made up of shifted versions of $F(\omega)$ (aliased) defined in ((9)) Because ((8)) and ((6)) are equal for all Tn and because the limits can be shifted by π/T without changing the equality, the integrands are equal and we have

$$F_d(\omega) = \frac{1}{T} \sum_\ell F(\omega + 2\pi\ell/T) = \frac{1}{T} F_p(\omega). \tag{9}$$

where $F_p(\omega)$ is a periodic function made up of shifted versions of $F(\omega)$ as in ((8)). The spectrum of the samples of $f(t)$ is an aliased version of the spectrum of $f(t)$ itself. The closer together the samples are taken, the further apart the centers of the aliased spectra are.

This result is very important in determining the frequency domain effects of sampling. It shows what the sampling rate should be and it is the basis for deriving the sampling theorem. Samples of the Spectrum of a Sampled Signal and the DFT

Samples of the spectrum can be calculated from a finite number of samples of the original continuous-time signal using the DFT. If we let the length of the DFT be N and separation of the samples in the frequency domain be Δ and define the periodic functions

$$F_p(\omega) = \sum_\ell F(\omega + N\Delta\ell) \tag{10}$$

and

$$f_p(t) = \sum_m f(t + NTm) \tag{11}$$

then from ((5)) and ((9)) samples of the DTFT of $f(Tn)$ are

$$F_p(\Delta k) = T\sum_n f(Tn)\, e^{-jT\Delta nk}$$

$$= T\sum_m \sum_{n=0}^{N-1} f(Tn+TNm)\, e^{-j\Delta(Tn+TNm)k}$$

$$= T\sum_{n=0}^{N-1}\left[\sum_m f(Tn+TNm)\right] e^{-j\Delta(Tn+TNm)k}, \tag{12}$$

therefore,

$$F_p(\Delta k) = \mathscr{DFT}\{f_p(Tn)\} \tag{13}$$

if $\Delta TN = 2\pi$. This formula gives a method for approximately calculating values of the Fourier transform of a function by taking the DFT (usually with the FFT) of samples of the function. This formula can easily be verified by forming the Riemann sum to approximate the integrals in ((1)) or ((2)).

Samples of the DTFT of a Sequence

It the signal is discrete in origin and is not a sampled function of a continuous variable, the DTFT is defined with $T = 1$ as

$$H(\omega) = \sum_n h(n)\, e^{-j\omega n} \tag{14}$$

with an inverse

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)\, e^{j\omega n} d\omega. \tag{15}$$

If we want to calculate $H(\omega)$, we must sample it and that is written as

$$H(\Delta k)\sum_n h(n)\, e^{-j\Delta kn}$$

which after breaking the sum into an infinite sum of length-N sums as was done in ((12)) becomes

$$H(\Delta k)\sum_m \sum_{n=0}^{N-1} h(n+Nm)\, e^{-j\Delta kn}$$

if $\Delta = 2\pi/N$. This allows us to calculate samples of the DTFT by taking the DFT of samples of a periodized $h(n)$.

$$H(\Delta k) = \mathscr{DFT}\{h_p(n)\}. \tag{16}$$

This a combination of the results in ((9)) and in ((13)).

Fourier Series Coefficients from the DFT

If the signal to be analyzed is periodic, the Fourier integral in ((1)) does not converge to a function (it may to a distribution). This function is usually expanded in a Fourier series to define its spectrum or a frequency description. We will sample this function and show how to approximately calculate the Fourier series coefficients using the DFT of the samples.

Consider a periodic signal $\tilde{f}(t) = \tilde{f}(t+P)$ with N samples taken every T seconds to give $\tilde{Tn}(t)$ for integer n such that $NT = P$. The Fourier series expansion of $\tilde{f}(t)$ is

$$\tilde{f}(t) = \sum_{k=-\infty}^{\infty} C(k)\, e^{2\pi kt/P} \tag{17}$$

with the coefficients given in ((3)). Samples of this are

$$\tilde{f}(Tn) = \sum_{k=-\infty}^{\infty} C(k) e^{2\pi kTn/P} = \sum_{k=-\infty}^{\infty} C(k) e^{2\pi kn/N} \tag{18}$$

which is broken into a sum of sums as

$$\tilde{f}(Tn) = \sum_{\ell-\infty}^{\infty} \sum_{k=0}^{N-1} C(k+N\ell) e^{2\pi(k+N\ell)n/N} = \sum_{k=0}^{N-1} \left[\sum_{\ell-\infty}^{\infty} C(k+N\ell) \right] e^{2\pi kn/N}. \tag{19}$$

But the inverse DFT is of the form

$$\tilde{f}(Tn) = \frac{1}{N} \sum_{k=0}^{N-1} F(k) e^{j2\pi nk/N} \tag{20}$$

therefore,

$$\mathscr{DFT}\{\tilde{f}(Tn)\} = N \sum_{\ell} C(k+N\ell) = NC_p(k). \tag{21}$$

and we have our result of the relation of the Fourier coefficients to the DFT of a sampled periodic signal. Once again aliasing is a result of sampling.

Shannon's Sampling Theorem

Given a signal modeled as a real (sometimes complex) valued function of a real variable (usually time here), we define a bandlimited function as any function whose Fourier transform or spectrum is zero outside of some finite domain

$$|F(\omega)| = 0 \text{[U+EEF2] [U+EEF2]for [U+EEF2] [U+EEF2]} |\omega| > W \tag{22}$$

for some $W < \infty$. The sampling theorem states that if $f(t)$ is sampled

$$f_s(n) = f(Tn) \tag{23}$$

such that $T < 2\pi/W$, then $f(t)$ can be exactly reconstructed (interpolated) from its samples $f_s(n)$ using

$$f(t) = \sum_{n=-\infty}^{\infty} f_s(n) \left[\frac{\sin(\pi t/T - \pi n)}{\pi t/T - \pi n} \right]. \tag{24}$$

This is more compactly written by defining the sinc function as

$$\operatorname{sinc}(x) = \frac{\sin(x)}{x} \tag{25}$$

which gives the sampling formula () the form

$$f(t) = \sum_{n} f_s(n) \operatorname{sinc}(\pi t/T - \pi n). \tag{26}$$

The derivation of () or () can be done a number of ways. One of the quickest uses infinite sequences of delta functions and will be developed later in these notes. We will use a more direct method now to better see the assumptions and restrictions.

We first note that if $f(t)$ is bandlimited and if $T < 2\pi/W$ then there is no overlap or aliasing in $F_p(\omega)$. In other words, we can write ((2)) as

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} F_p(\omega) e^{j\omega t} d\omega \tag{27}$$

but

$$F_p(\omega) = \sum_{\ell} F(\omega + 2\pi\ell/T) = T \sum_{n} f(Tn)\, d^{-j\omega Tn} \tag{28}$$

therefore,

$$f(t) = \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \left[T \sum_{n} f(Tn)\, e^{-j\omega Tn} \right] e^{j\omega t} d\omega \tag{29}$$

$$= \frac{T}{2\pi} \sum_{n} f(Tn) \int_{-\pi/T}^{\pi/T} e^{j(t-Tn)\omega} d\omega$$

$$= \sum_{n} f(Tn) \frac{\sin\left(\frac{\pi}{T}t - \pi n\right)}{\frac{\pi}{T}t - \pi n} \tag{30}$$

which is the sampling theorem. An alternate derivation uses a rectangle function and its Fourier transform, the sinc function, together with convolution and multiplication. A still shorter derivation uses strings of delta function with convolutions and multiplications. This is discussed later in these notes.

There are several things to notice about this very important result. First, note that although $f(t)$ is defined for all t from only its samples, it does require an infinite number of them to exactly calculate $f(t)$. Also note that this sum can be thought of as an expansion of $f(t)$ in terms of an orthogonal set of basis function which are the sinc functions. One can show that the coefficients in this expansion of $f(t)$ calculated by an inner product are simply samples of $f(t)$. In other words, the sinc functions span the space of bandlimited functions with a very simple calculation of the expansion coefficients. One can ask the question of what happens if a signal is “under sampled". What happens if the reconstruction formula in () is used when there is aliasing and () is not true. We will not pursue that just now. In any case, there are many variations and generalizations of this result that are quite interesting and useful.

References

[1] R. N. Bracewell. *The Fourier Transform and Its Applications.* McGraw-Hill, New York, third edition, 1985.

[2] A. Papoulis. *The Fourier Integral and Its Applications.* McGraw-Hill, 1962.

[3] R. M. Young. *An Introduction to Nonharmonic Fourier Series.* Academic Press, New York, 1980.