Laws governing collision*

Sunil Kumar Singh

Abstract

Foremost among the laws governing collsion is the conservation of linear momentum as there is no external force involved.

Collision, as we know, is an event of very short time interval. It is imperative that we look for some mechanism that enables us to predict the result subsequent to collision without knowing what happens during collision. Fortunately, there are simple underlying principles that govern collision process. In this module, we shall discuss governing laws for different situation pertaining to collision.

1 Conservation of linear momentum

Foremost among the laws governing collision is the conservation of linear momentum as there is no external force involved or the external force is small enough with respect to collision force and can be neglected without affecting the result in any appreciable manner.

For the system of colliding bodies, linear momentum of the system is same before and after collision :

$$\mathbf{P}_i = \mathbf{P}_f$$

For two colliding bodies, the above conservation law can be written as :

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Conservation of linear momentum

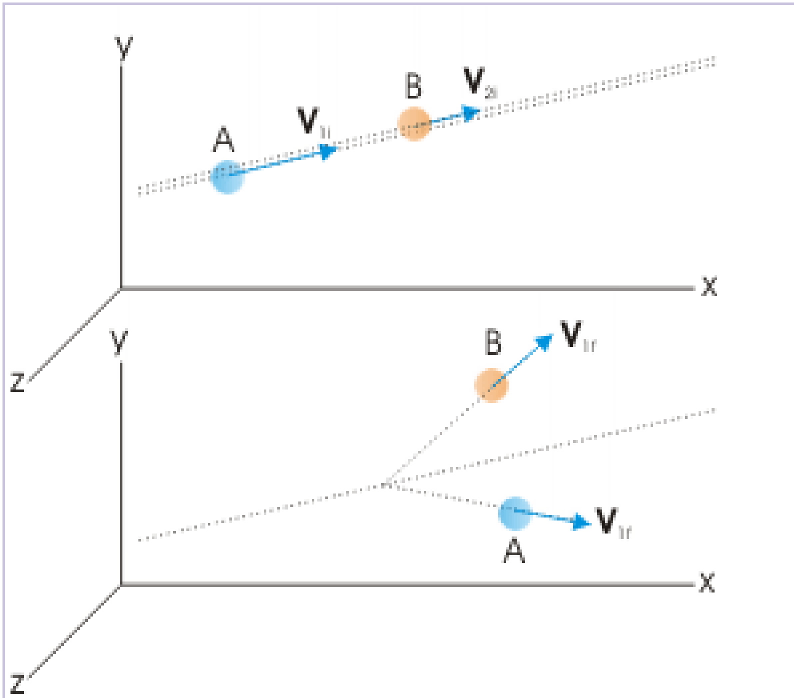

Figure 1: The bodies collide to galnce off in different direction.

$$m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = m_1\mathbf{v}_{1f} + m_2\mathbf{v}_{2f}$$

Suffix "1" and "2" refer to the two colliding bodies, whereas suffix "i" and "f" refer to the "initial" and "final" states (before and after) of the collision respectively. It is important to visualize that two bodies will collide when their trajectory is such that the physical dimension of the bodies overlap. Further, one of the bodies approaches second body with greater speed ($v_{1i} > v_{2i}$) for collision to occur. In the figure above, two bodies are shown to have parallel trajectories so that body "A" does not strike "B" head on, but glances to move in different directions after collision.

Generally, the body with greater velocity that collides with another body is conventionally termed as "projectile" and the body which is hit is termed as "target".

We must understand that the law of conservation of linear momentum is valid for a closed system, wherein there is no exchange of mass between the system and its surrounding. Also, we must note that above mathematical construct is a vector equation. This can be expressed in component form as :

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$
$$m_1 v_{1iy} + m_2 v_{2iy} = m_1 v_{1fy} + m_2 v_{2fy}$$
$$m_1 v_{1iz} + m_2 v_{2iz} = m_1 v_{1fz} + m_2 v_{2fz}$$

When bodies collide head-on, they move along a straight line. Such collision in one dimension can be handled with any of the component equations. For convenience, we can drop the component suffix and write the conservation law without suffix :

Collision in one dimension

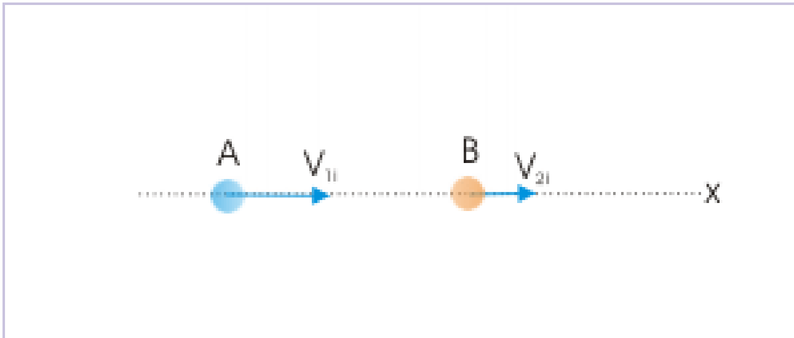

Figure 2: Head on collision results motion in one diemsion.

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

As a reminder, we need to emphasize here that consideration of conservation of linear momentum is independent of following important aspects of collision :

- Collision force or time of collision
- Nature of collision : elastic or inelastic
- Dimension of collision : one, two or three

1.1 Conservation of linear momentum in completely inelastic collision

Completely inelastic collision is a special case of inelastic collision. In this case, the colliding body is completely embedded into target after collision. The two colliding bodies thereafter move with same velocity. Let the common speed be V such that :

Completely inelastic collision

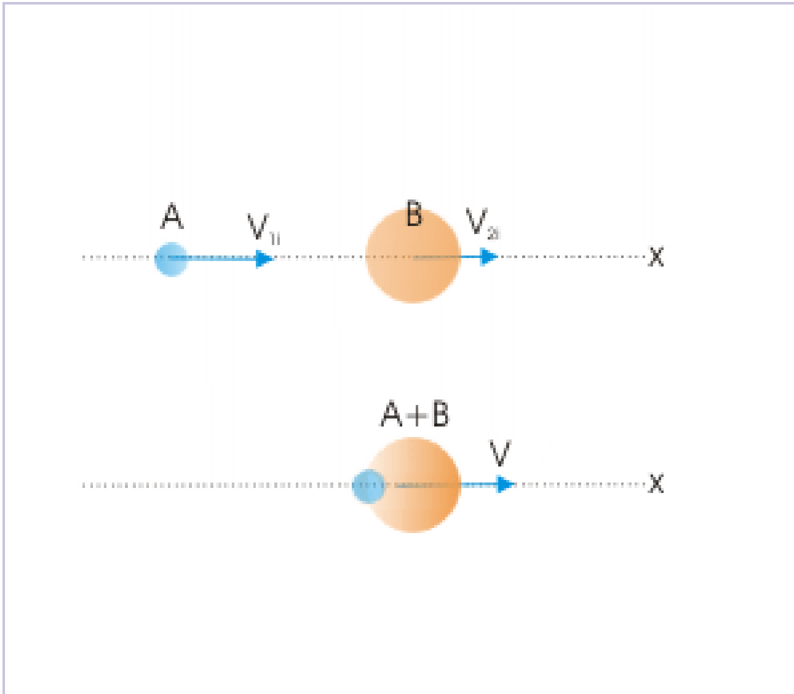

Figure 3: The colliding move with same velocity.

$$\mathbf{v}_{1f} = \mathbf{v}_{2f} = \mathbf{V}$$

Putting this in to the equation of linear momentum, we have :

$$m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = m_1\mathbf{V} + m_2\mathbf{V} = (m_1 + m_2)\,\mathbf{V}$$

If the collision is one dimensional also, then we can drop vector notation :

$$m_1 v_{1i} + m_2 v_{2i} = m_1 V + m_2 V = (m_1 + m_2)\,V$$

Example 1

Problem : A ballistic pendulum comprising of wooden block of 10 kg is hit by a bullet of 10 gm in horizontal direction. The bullet quickly comes to rest as it is embedded into the wooden block. If the wooden block and embedded bullet moves with a speed of 1.414 m/s after collision, then find the speed of bullet.

Ballistic pendulum

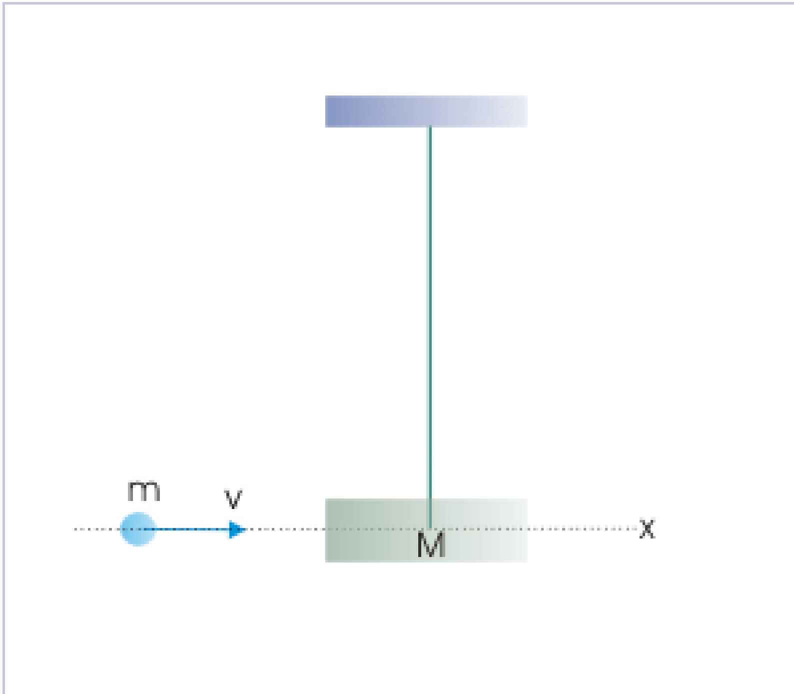

Figure 4: The bullet and block undergo completely inelastic collision.

Solution : The collision lasts for short period. During collision, the combined mass of the block and bullet is subjected to gravitational and tension forces, which balance each other. As such, the system of colliding bodies is subjected to zero external force. The governing principle, in this case, is conservation of linear momentum. Now, as the bullet sticks to the wooden block, the collision is completely inelastic.

Let m and M be the masses of bullet and block respectively. Let us also consider that bullet strikes the block with a velocity "v" and the block and embedded bullet together move with the velocity "V" after the collision. We assume that bullet hits the block head on and the combined mass of block and bullet moves in the horizontal direction just after the collision. Hence, applying conservation of linear momentum in one dimension, we have :

$$(m + M) V = mv$$
$$\Rightarrow v = \frac{m + M}{m} V$$
$$\Rightarrow v = \frac{0.01 + 10}{0.01} x\ 1.414 = 1415\, m / s$$

2 Center of mass and collision

The fact that there is no external force during collision has important bearing on the motion of center of mass. The motion of the center of mass can not change in the absence of external force; internal forces appearing during the collision can not change the velocity of center of mass. As such, motion of the center of mass remains unaffected after collision. In other words, acceleration of center of mass is zero. Mathematically :

$$\mathbf{v}_{\text{com}} = \text{constant}$$
$$\mathbf{a}_{\text{com}} = 0$$

We can combine this fact with the conservation law to obtain the expression of center of mass. The linear momentum of the system of two colliding bodies in terms of center of mass is :

$$M\mathbf{v}_{\text{com}} = m_1\mathbf{v}_{1\text{i}} + m_2\mathbf{v}_{2\text{i}} = m_1\mathbf{v}_{1\text{f}} + m_2\mathbf{v}_{2\text{f}}$$

$$\Rightarrow \mathbf{v}_{\text{com}} = \frac{m_1\mathbf{v}_{1\text{i}} + m_2\mathbf{v}_{2\text{i}}}{m_1 + m_2} = \frac{m_1\mathbf{v}_{1\text{f}} + m_2\mathbf{v}_{2\text{f}}}{m_1 + m_2}$$

This vector equation can be equivalently written into three scalar component equations. For one dimensional case, however, we drop the vector notation altogether as in the case of linear momentum :

$$\Rightarrow v_{\text{com}} = \frac{m_1 v_{1\text{i}} + m_2 v_{2\text{i}}}{m_1 + m_2} = \frac{m_1 v_{1\text{f}} + m_2 v_{2\text{f}}}{m_1 + m_2}$$

It must be understood that these relations are general relation like that of linear momentum. Again as a reminder, we need to emphasize here that consideration of center of mass for collision is independent of following important aspects of collision :

- Collision force or time of collision
- Nature of collision : elastic or inelastic
- Dimension of collision : one, two or three

2.1 Center of mass and completely inelastic collision

As in the case of conservation of linear momentum in one dimension, the completely inelastic collision provides the unique condition when,

$$v_{1\text{f}} = v_{2\text{f}} = v$$

$$\Rightarrow v_{\text{com}} = \frac{m_1 V + m_2 V}{m_1 + m_2} = V$$

This result is expected also as two bodies is converted into one. The center of mass, therefore, has the same velocity as that of the combined (or stuck) mass.

3 Kinetic energy and collision

Unlike linear momentum, kinetic energy is not conserved in all collisions. This depends on the type of collision i.e. whether the collision is elastic or inelastic. Here, we shall discuss these cases.

> NOTE: Like kinetic energy, the potential energy of the system before and after collision is not considered for analyzing collision. It is so because, we concentrate on the velocities just before and after the collision, which occurs in very short period. There may not be any change in the elevation of colliding bodies before and after collision. For this reason, we analyze collision even along an incline with out considering potential energy of the colliding bodies as far as immediate states "before" and "after" are concerned.

3.1 Kinetic energy in elastic collision

In elastic collision, the colliding bodies get deformed to different degree depending on the nature of material composing them. The kinetic energy of the system is temporarily transferred into elastic potential energy, which is regained subsequently. The nature of bodies, however, determine the time for which colliding bodies are in contact. Harder the bodies, smaller is the time of contact and vice-versa. The collision between two hard bodies is likely to produce elastic collision. As kinetic energy is regained at the end of collision, kinetic energy is conserved in elastic collision. Thus,

$$\mathbf{K}_i = \mathbf{K}_f$$

For two colliding bodies, the above conservation law can be written as :

$$\frac{1}{2} m_1 v_{1\mathrm{i}}^2 + \frac{1}{2} m_2 v_{2\mathrm{i}}^2 = \frac{1}{2} m_1 v_{1\mathrm{f}}^2 + \frac{1}{2} m_2 v_{2\mathrm{f}}^2$$

It is important to realize that this relation for elastic collision is an additional relation, which can be used in conjunction with the relation for conservation of linear momentum. We must also note that kinetic energy is a scalar quantity unlike linear momentum. This has one important implication. Irrespective of the dimension of motion, there is no component form of relation for kinetic energy. In all case, there is only one relation of kinetic energy as given above (compare this with three component equation for conservation of momentum).

3.2 Kinetic energy in inelastic collision

In inelastic collision, kinetic energy of the system is not conserved. Some of the kinetic energy is irrevocably transferred to the other forms of energy like sound or heat energy. As a result,

$$\mathbf{K}_i > \mathbf{K}_f$$

For two colliding bodies, the relation for kinetic energy can be written as :

$$\frac{1}{2} m_1 v_{1\mathrm{i}}^2 + \frac{1}{2} m_2 v_{2\mathrm{i}}^2 > \frac{1}{2} m_1 v_{1\mathrm{f}}^2 + \frac{1}{2} m_2 v_{2\mathrm{f}}^2$$

It is evident that this relationship of inequality does not help to evaluate motion of bodies after collision as the same can not be used in conjunction with momentum equation.

The loss of energy in the collision is estimated taking the difference of kinetic energy before and after the collision.

$$\Delta K = K_i - K_f = \frac{1}{2} \left(m_1 v_{1\mathrm{i}}^2 + m_2 v_{2\mathrm{i}}^2 - m_1 v_{1\mathrm{f}}^2 - m_2 v_{2\mathrm{f}}^2 \right)$$

If loss of kinetic energy (ΔK) can be determined experimentally, then it is possible to render inequality into an equality as :

$$K_i = K_f + \Delta K$$

For two colliding bodies, the conservation of kinetic energy can be written as :

$$\frac{1}{2} m_1 v_{1\mathrm{i}}^2 + \frac{1}{2} m_2 v_{2\mathrm{i}}^2 = \frac{1}{2} m_1 v_{1\mathrm{f}}^2 + \frac{1}{2} m_2 v_{2\mathrm{f}}^2 + \Delta K$$

Now, we can use this equation in conjunction with the equations of linear momentum to determine the unknown attributes of collision.

Example 2

Problem : A block of mass "m" moving at speed "v" collides with a stationary block of mass "2m" head-on. If the lighter block comes to a stop immediately after the collision, then determine

(i) whether the collision is elastic or inelastic and (ii) loss of kinetic energy to the surrounding, if any.

Solution : It is a one dimensional collision. Applying conservation of linear momentum in one dimension,

$$m_1 v_{1\mathrm{i}} + m_2 v_{2\mathrm{i}} = m_1 v_{1\mathrm{f}} + m_2 v_{2\mathrm{f}}$$

Here, $m_1 = m$, $m_2 = 2\mathrm{m}$, $v_{1\mathrm{i}} = v$, $v_{2\mathrm{i}} = 0$. Let the heavier block moves with a velocity "V" after the collision. Then,

$$mv + 2\mathrm{m}\ x\ 0 = m\ x\ 0 + 2\mathrm{mV}$$
$$\Rightarrow V = \frac{v}{2}$$

Now, kinetic energy of the system of two colliding bodies before collision is :

$$K_i = \frac{1}{2} m v^2$$

and the kinetic energy of the system of two colliding bodies after collision is :

$$K_f = \frac{1}{2} m V^2 = \frac{1}{2} m \left(\frac{v}{2} \right)^2 = \frac{mv^2}{8}$$

Thus, $K_i > K_f$. Hence, collision is inelastic. The loss of kinetic energy to the surrounding is :

$$\Delta K = K_i - K_f = \frac{mv^2}{2} - \frac{mv^2}{8} = \frac{3mv^2}{8}$$

3.2.1 Kinetic energy in completely inelastic collision

The completely inelastic collision is a subset of inelastic collision. It provides the unique condition when final kinetic energy pertains to the combined mass. Let us consider that collision takes place in one dimension and the common velocity of the embedded body is V. Then,

$$\mathbf{v}_{1\mathrm{f}} = \mathbf{v}_{2\mathrm{f}} = \mathbf{V}$$

Kinetic energy inequality is written as :

$$\frac{1}{2} m_1 v_{1\mathrm{i}}^2 + \frac{1}{2} m_2 v_{2\mathrm{i}}^2 > \frac{1}{2} \left(m_1 + m_2 \right) V^2$$

Example 3

Problem : A ballistic pendulum comprising of wooden block of 10 kg is hit by a bullet of 10 gm in horizontal direction. The bullet quickly comes to rest as it is embedded into the wooden block. If the wooden block rises by a height 10 cm as it swings up, then find the speed of the bullet.

Ballistic pendulum

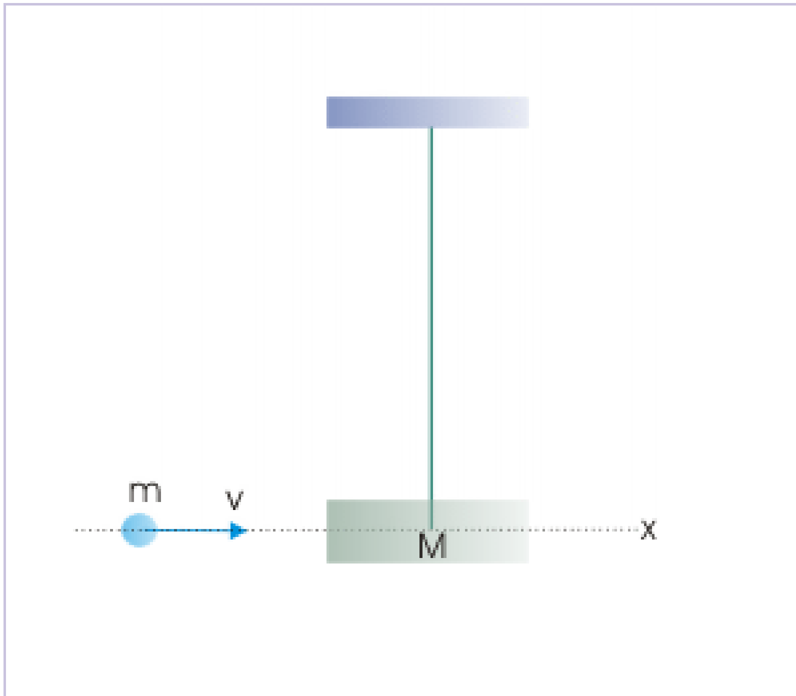

Figure 5: The bullet and block undergo completely inelastic collision.

Solution : We find the relation of velocities using conservation of linear momentum as discussed in the first example of this module.

$$(m + M)V = mv$$
$$\Rightarrow V = \frac{mv}{m + M}$$

The combined mass of the bullet and block rises to a vertical height "h" due to swing. In this process, the kinetic energy of the block with embedded bullet is converted into the change in potential energy of the system. Hence,

Ballistic pendulum

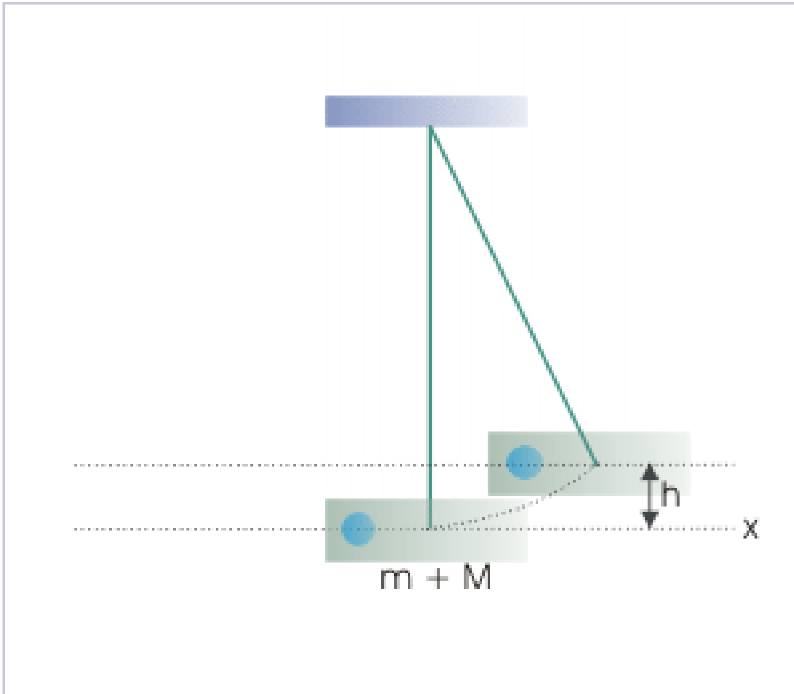

Figure 6: Kinetic energy is converted into gravitational potential energy as the bullet and block system rises as .

$$\frac{1}{2}\ (\ m\ +\ M\ )\ V^2\ =\ (\ m\ +\ M\ )\ gh$$
$$\Rightarrow\ \frac{1}{2}V^2\ =\ gh$$

Substituting for "V" from the first equation,

$$\frac{m^2 v^2}{2\ (\ m + M\ )\ ^2}\ =\ gh$$
$$\Rightarrow\ v\ =\ \frac{m + M}{m}\ \surd\ (\ 2gh\ )$$

Putting values, we have :

$$v\ =\ \frac{0.01 + 10}{0.01}\ x\ \surd\ (\ 2\ x\ 10\ x\ 0.1\ )\ =\ 1415\, m\ /\ s$$