Pulley in rolling*

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Abstract

Motion of a pulley of finite mass is analogous to rolling.

In our earlier treatment on "pulley" in this course, we had limited our consideration to "mass-less" pulley. Here, we shall consider pulley, which has finite mass and is characterized by rolling motion and presence of friction in certain cases.

We need to distinguish two situations (finite mass and mass-less cases) in order to analyze the motion correctly. Mass-less pulley is characterized by the fact that it does not affect the magnitude of tension in the string. It means that tensions in the string on either side of the pulley remains same. In general, a "mass-less" pulley changes the direction of force (tension) without any change in magnitude.

1 Analysis of the motion of Pulley in rolling

A pulley of finite mass, on the other hand, may rotate, fulfilling the condition of rolling. In this case, the length of rope/string released from the pulley is equal to the distance covered by a point on the rim. If the rolling is accelerated, there is friction between the pulley surface and the string/ rope, passing over it, enabling the pulley to accelerate/decelerate in rotation.

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Pulley

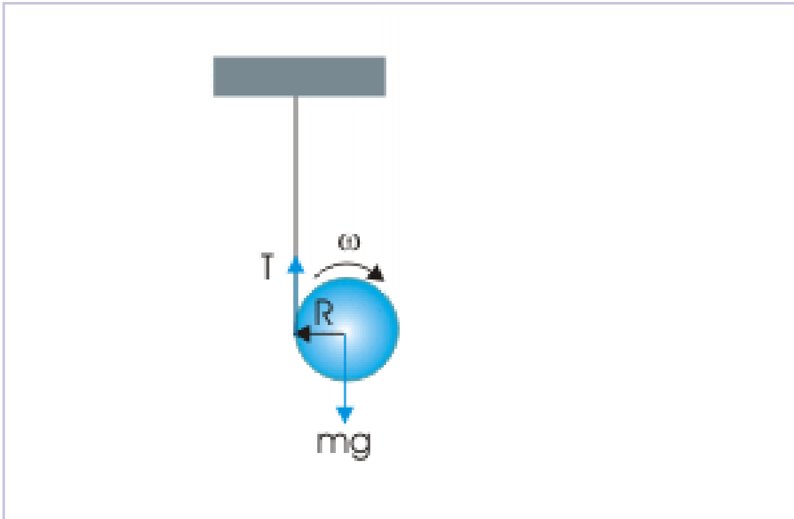

Figure 1: Pulley is rolling on the rope/ string in vertically downward direction.

Evidently, the acceleration of pulley of finite mass will be associated with a net force on the pulley. This, in turn, means that the tension in the strong are not same as in the case of "mass-less" pulley. The pulley, in the figure above, translates and rotates with acceleration, as the string wrapped over it unwinds.

The length of rope unwound is equal to the vertical distance traveled by the pulley/ disk as in the case of rolling and as shown in the figure. Hence,

Pulley in rolling

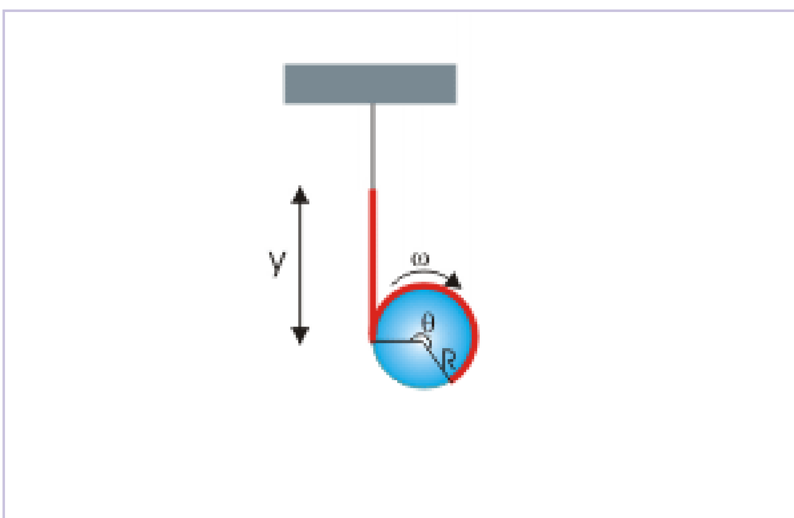

Figure 2: The length of rope unwound is equal to the vertical distance traveled by the pulley.

$$y = \theta R$$

The motion of pulley may not exactly look like rolling. But, we can see that string/rope plays the role of a surface in rolling. This analogy is not very obscure as string provides a tangential surface like horizontal surface for the pulley to roll. This is an analogous situation to the rolling of a disk. Differentiating the relation, as given above, with respect to time :

$$v_C = \omega R \quad (1)$$

$$a_C = \alpha R \quad (2)$$

In certain situation, the pulley may be fixed to the ceiling as shown in the figure below and hence incapable of translation. We can not say here that pulley is actually not rolling. But, the rope translates as much as a point on the rim of the pulley and as such the rope translates at the same velocity and acceleration as that of the center of mass of the pulley, if it were free to translate. We can see that pulley is executing the rotational part of the rolling motion, whereas string, along with attached blocks, is executing the translational part of the rolling motion. Thus, motions of pulley and string together are equivalent to rolling motion.

Pulley

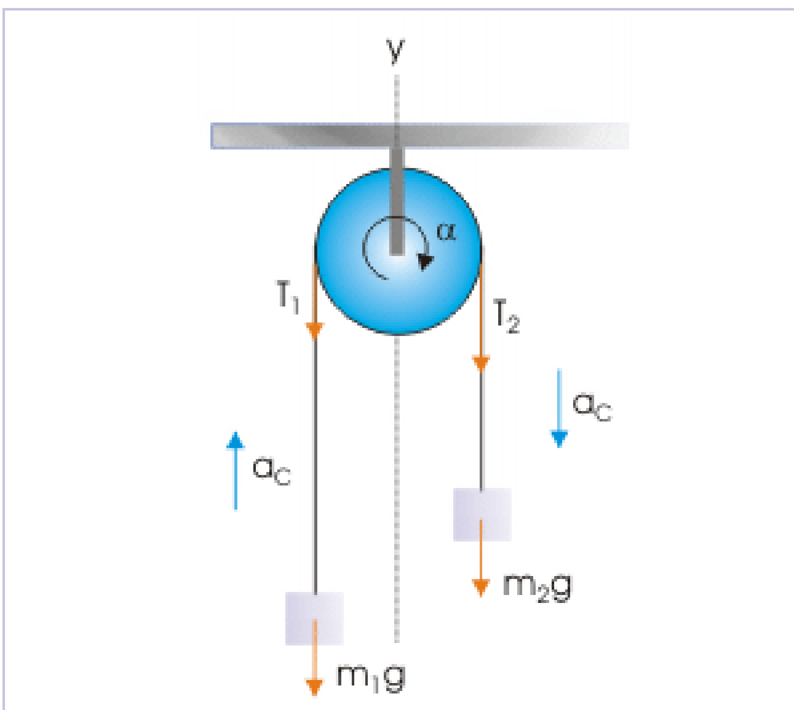

Figure 3: Fixed pulley and string together executes rolling.

We analyze motion of pulley in same manner as that of a rolling body with the help of two Newton's second laws – one for the linear motion and other for the angular motion. Such consideration of law of motion, however, is conditioned by the equation of rolling and equation of accelerated rolling.

2 Examples

In this section, we work out with few representative examples to illustrate the application of rolling motion in the case of pulley :

- Pulley or disk of finite mass is translating
- Pulley or disk of finite mass is stationary
- A mass-less pulley connects rolling motion of a disk to the translation of a block.

2.1 Pulley or disk of finite mass is translating

Example 1

Problem : A long rope of negligible mass is wrapped many times over a solid cylinder of mass "m" and radius "R". Other end of the rope is attached to a fixed ceiling and the cylinder is then let go at a given instant. Find the tension in the string and acceleration of the cylinder.

Rolling down the rope

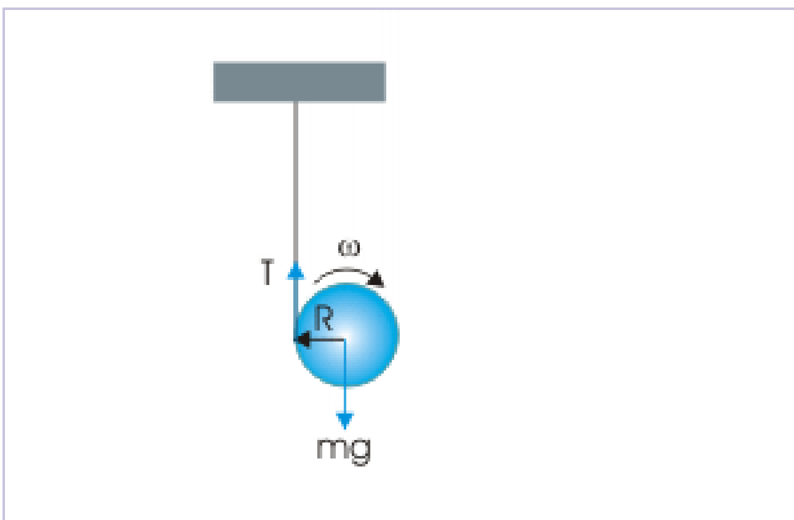

Figure 4: The rolling of the cylinder is guided on the rope.

Solution : The rolling of the cylinder is guided on the rope. We select a coordinate system in which positive y-direction is along the downward motion as shown in the figure.

Rolling down the rope

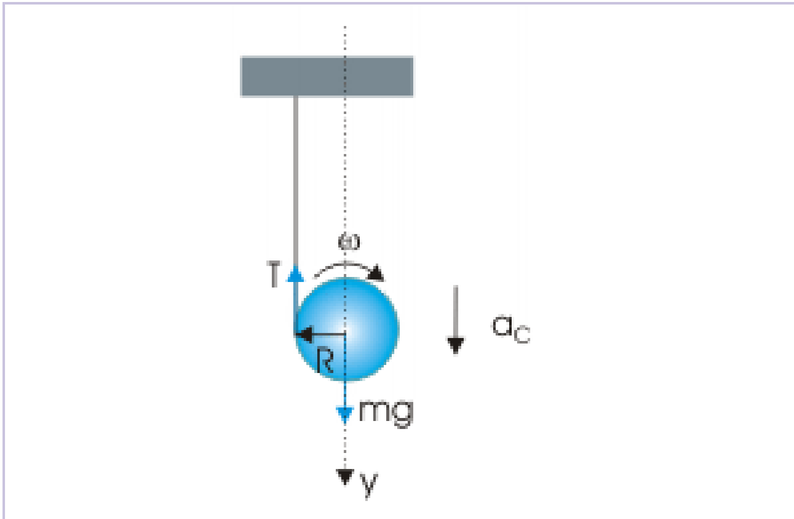

Figure 5: Free body diagram.

As discussed earlier, we set out to write three equations and solve for the required quantity. Three equations for rolling motions are (i) Newton's second law for translation (ii) Newton's second law for rotation and (iii) Equation of accelerated rolling.

From the application of Newton's second law for translation in y - direction :

$$\sum F_y = \text{mg} - T = ma_C \tag{3}$$

From the application of Newton's second law for rotation :

$$\tau = TR = I\alpha$$

$$T = \frac{I\alpha}{R} \tag{4}$$

Note that force due to gravity passes through center of mass and does not constitute a torque to cause angular acceleration. Also, we do not consider sign for angular quantities as we shall be using only the magnitudes here . Now, from relation of linear and angular accelerations (equation of accelerated rolling), we have :

$$a_C = \alpha R \tag{5}$$

Putting the value of "α" from above in the equation - 4, we have :

$$T = I\frac{a_C}{R^2}$$

The moment of inertia for solid cylinder about its axis is,

$$I = \frac{mR^2}{2}$$

Hence,

$$\Rightarrow T = \frac{mR^2 a_C}{2R^2}$$
$$\Rightarrow T = \frac{ma_C}{2} \tag{6}$$

Substituting the expression of tension in the equation of force analysis in y-direction (equation - 3), we have :

$$\text{mg} - \frac{ma_C}{2} = ma_C$$
$$\Rightarrow g - \frac{a_C}{2} = a_C$$
$$\Rightarrow a_C \left(1 + \frac{1}{2}\right) = g$$
$$\Rightarrow a_C = \frac{2g}{3}$$

Putting this value in the expression of "T" (equation - 6), we have :

$$\Rightarrow T = \frac{ma_C}{2} = \frac{mg}{3}$$

2.2 Pulley or disk of finite mass is stationary

Example 2

Problem : In the "pulley – blocks" arrangement shown in the figure. The masses of the pulley and two blocks are "M", " m_1 " and " m_2 " respectively. If there is no slipping between pulley and rope, then find (i) acceleration of the blocks and (ii) tensions in the string.

Pulley – blocks system

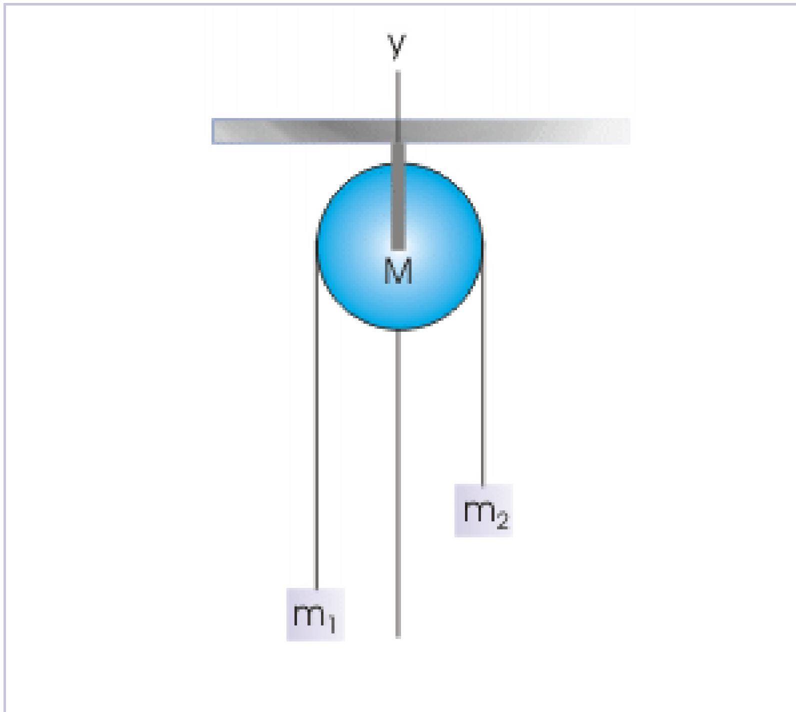

Figure 6: There is no slipping between pulley and rope.

Solution : The blocks have different masses (hence weights). The net force on the pulley due to difference in weight accelerates the pulley in rotation. Let the tensions in the string be " T_1 " and " T_2 ". Since the rope is inextensible, the different points on the rope has same acceleration as that of the rolling i.e. a_C . Let $m_2 > m_1$.

Pulley – blocks system

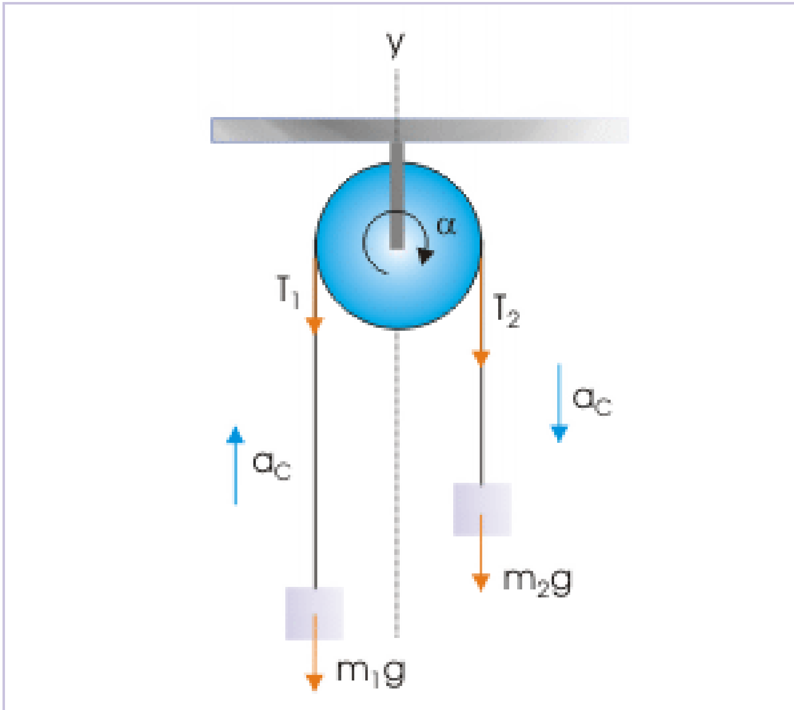

Figure 7: There is no slipping between pulley and rope.

The magnitude of torque on the pulley (we neglect the consideration of direction) is :

$$T = (T_2 - T_1) \; x \; R$$

The pulley is fixed to the ceiling. As such, it is constrained not to translate in response to the net vertical force on it. Instead, the net vertical force causes translational acceleration of the string and the masses attached to the string. The length of string released from the pulley is equal to the distance covered by a point on the rim of the pulley. It means that the linear acceleration of the string is related to angular acceleration of the pulley by the equation of accelerated rolling. Hence,

$$a_C = \alpha R$$

The directions of angular and linear accelerations are as shown in the figure. Now, considering pulley and blocks, we have three equations as :

(i) Rotation of pulley :

$$(T_2 - T_1) \; x \; R = I\alpha = \frac{1}{2} \, x \, MR^2\alpha$$

$$\Rightarrow \alpha = \frac{2 \, (T_2 - T_1)}{MR}$$

and

$$\Rightarrow a_C = \alpha R = \frac{2 \, (T_2 - T_1)}{M} \tag{7}$$

(ii) Translation of m_1 :

$$T_1 \ - \ m_1 g \ = \ m_1 a_C \tag{8}$$

(iii) Translation of m2 :

$$m_2 g \ - \ T_2 \ = \ m_2 a_C \tag{9}$$

Putting values of " T_1 " and " T_2 " from equations 7 and 8, in the expression of acceleration (equation - 6), we have :

$$\Rightarrow a_C \ = \ \frac{2 \ (\ m_2 g - m_2 a_C - m_1 g - m_1 a_C \ )}{M} \tag{10}$$

Reaaranging,

$$\Rightarrow a_C \ (\ M \ + \ 2m_1 \ + \ 2m_2 \ ) \ = \ 2g \ (\ 2m_2 \ - \ m_1 \ )$$

$$\Rightarrow a_C \ = \ \frac{2g \ (\ 2m_2 \ - \ m_1 \ )}{(\ M + \ 2m_1 \ + \ 2m_2 \ )}$$

Putting values of " a_C " from equation - 9, we have :

$$\Rightarrow T_1 \ = \ m_1 g \ + \ m_1 \ x \ \frac{2g \ (\ 2m_2 \ - \ m_1 \ )}{(\ M + \ 2m_1 \ + \ 2m_2 \ )}$$

Rearranging, we have :

$$\Rightarrow T_1 \ = \ \frac{m_1 g \ (\ M + \ 4m_2 \ )}{(\ M + \ 2m_1 \ + \ 2m_2 \ )}$$

Similarly,

$$\Rightarrow T_2 \ = \ \frac{m_2 g \ (\ M + \ 4m_1 \ )}{(\ M + \ 2m_1 \ + \ 2m_2 \ )}$$

2.3 A mass-less pulley connects rolling motion of a disk to the translation of a block

Example 3

Problem : In the "pulley – blocks" arrangement, a string is wound over a circular cylinder of mass "M". The string passes over a mass-less pulley as shown in the figure. The other end of the string is attached to a block of mass "m". If the cylinder is rolling on the surface towards right, then find accelerations of the cylinder and block.

Combination of motions via pulley

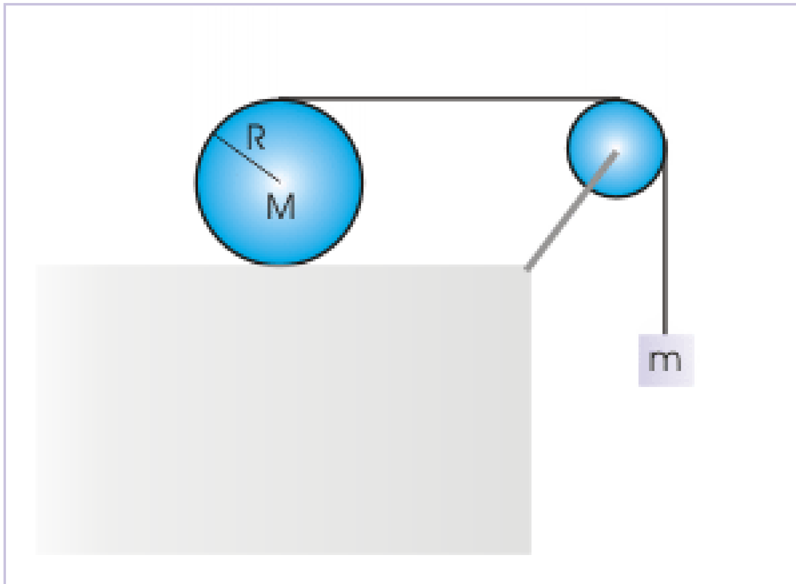

Figure 8: Rolling of cylinder and translation of block

Solution : In this question, string passes over a mass-less pulley. It means that the tension in the string through out is same. Since string is attached to the top of the cylinder, the tension force acts tangentially to the cylinder in rolling. In this case, the friction is acting in forward direction (as the cylinder tends to slide backward).

The linear acceleration of the top position and center of mass are different. As such, linear accelerations of the cylinder and blocks are different. Let the accelerations of cylinder and block be " a_1 "and " a_2 " respectively.

Combination of motions via pulley

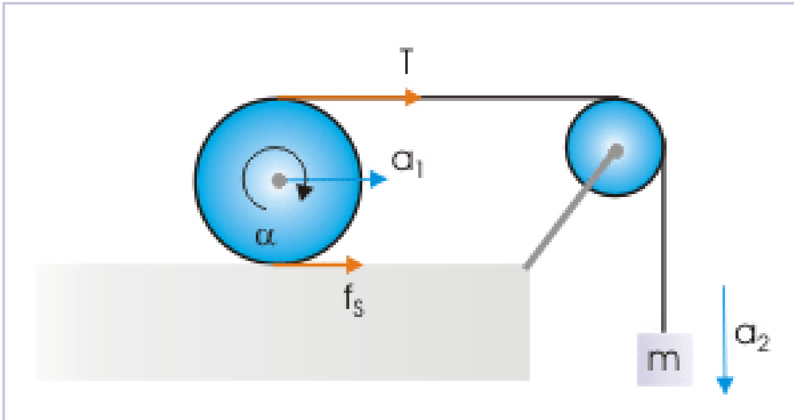

Figure 9: Rolling of cylinder and translation of block

(i) Rolling of cylinder :
The magnitude acceleration of the rolling cylinder (i.e. its COM) is :

$$a_C = a_1 = \alpha R \quad (11)$$

Now, we need to relate the acceleration of the cylinder to that of string (and hence that of block attached to it). For convenience, we consider only magnitudes here. Now, we know that the magnitude of the velocity of the top point of the cylinder is related to that of COM as :

$$v_T = 2v_C$$

Differentiating with respect to time, the magnitude of the acceleration of the string and the block, attached to it, is :

$$a_2 = a_T = 2a_C = 2a_1 = 2\alpha R \quad (12)$$

(ii) Translation of cylinder :
Applying Newton's second law for translation of the cylinder,

$$T + f_S = Ma_1 \quad (13)$$

(iii) Rotation of cylinder :
Applying Newton's second law for rotation,

$$(T - f_S) \ x \ R = I\alpha = \frac{1}{2} x \ MR^2\alpha$$

$$\Rightarrow \alpha = \frac{2 (T - f_S)}{MR}$$

$$\Rightarrow \alpha MR = 2T - f_S$$

For rolling, we use relation of accelerations as given by equation - 11 :

$$\Rightarrow 2T - 2f_S = Ma_1$$

Putting values of " f_S " from equation - 13,

$$\Rightarrow 2T - 2 (Ma_1 - T) = Ma_1$$

$$\Rightarrow T = \frac{3Ma_1}{4} \quad (14)$$

(iv) Translation of block :
The free body diagram of the block is shown in the figure. Here,

Free body diagram of block

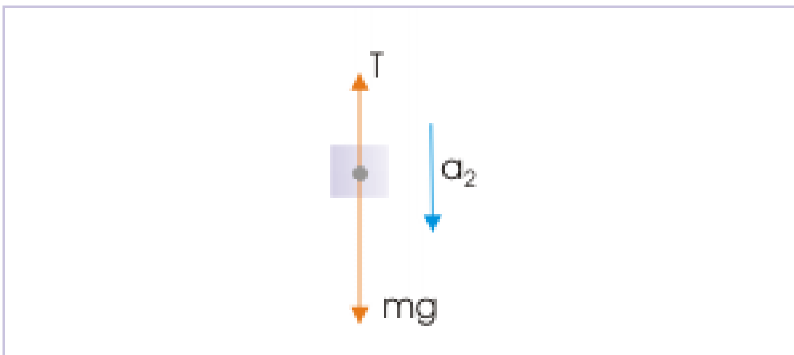

Figure 10: The block is accelerating down.

$$mg - T = ma_2$$

Substituting value of "T" from equation – 14,

$$ma_2 - mg = T = \frac{3Ma_1}{4}$$

Substituting this value of " a_1 " in terms of " a_2 " from equation – 11 and solving for " a_2 " ,

$$\Rightarrow a_2 = \frac{8mg}{(8m + 3M)}$$

As the magnitude of " a_1 " is given by :

$$\Rightarrow a_1 = \frac{a_2}{2} = \frac{4mg}{(8m + 3M)}$$