Angular momentum*

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Abstract

The angular momentum in rotation is a subset of angular momentum about a point in general motion.

Like linear momentum, angular momentum is the measure of the "quantity of motion". From Newton's second law, we know that first time derivative of linear momentum gives net external force on a particle. By analogy, we expect that this quantity (angular momentum) should have an expression such that its first time derivative yields torque on the particle.

1 Angular momentum about a point

Angular momentum is associated with a particle in motion. The motion need not be rotational motion, but any motion. Importantly, it is measured with respect to a fixed point.

Angular momentum of a particle about a point is defined as a vector, denoted as "ℓ".

$$\ell = \mathbf{r} \, \mathbf{x} \, \mathbf{p} \tag{1}$$

where "**r**" is the linear vector connecting the position of the particle with the "point" about which angular momentum is measured and "**p**" is the linear momentum vector. In case, the point coincides with the origin of coordinate system, the vector "**r**" becomes the position vector.

We should note here that small letter "ℓ" is used to denote angular momentum of a particle. The corresponding capital letter "L" is reserved for angular momentum of a system of particle or rigid body. This convention helps to distinguish the context and may be adhered to.

The SI unit of angular momentum is $\frac{\text{kg} - m^2}{s}$, which is equivalent to J-s.

1.1 Magnitude of angular momentum

Like in the case of torque, the magnitude of angular momentum can be obtained using any of the following relations :

*Version 1.4: Aug 26, 2009 4:00 am -0500

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Angular momentum of a particle

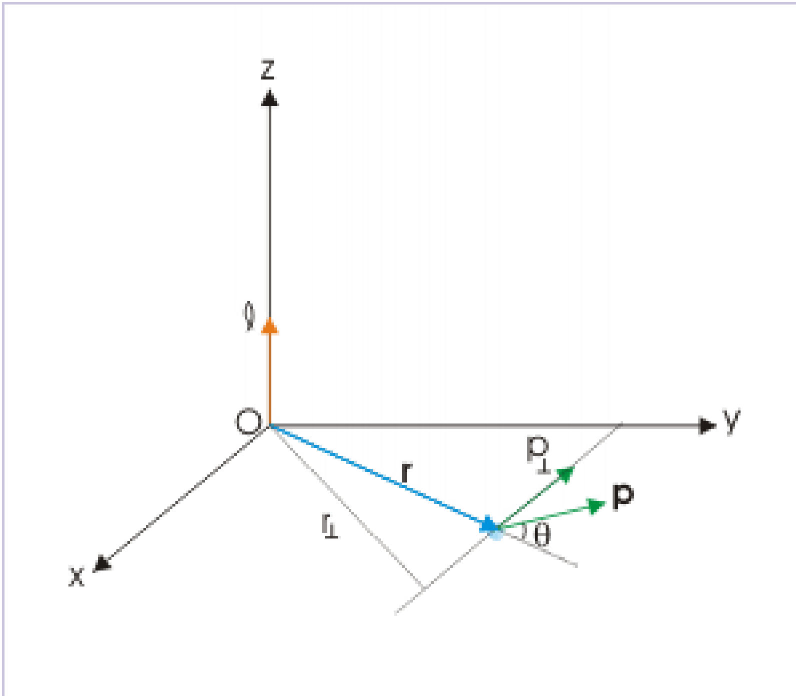

Figure 1: Angular momentum in terms of enclosed angle.

1: Angular momentum in terms of angle enclosed

$$\ell \;=\; rp\mathrm{sin}\theta \tag{2}$$

2: Angular momentum in terms of force perpendicular to position vector

$$\ell \;=\; rp_{\perp} \tag{3}$$

3: Angular momentum in terms of moment arm

$$\ell \;=\; r_{\perp}p \tag{4}$$

If the particle is moving with a velocity "**v**", then the expression of angular momentum becomes :

$$\ell \;=\; \mathbf{r\;x\;p} \;=\; m\;(\;\mathbf{r\;x\;v}\;) \tag{5}$$

Again, we can interpret this vector product as in the case of torque. Its magnitude can be obtained using any of the following relations :

$$\begin{aligned} \ell &= mrv\sin\theta \\ \ell &= mrv_{\perp} \\ \ell &= mr_{\perp}v \end{aligned} \quad (6)$$

Example 1

Problem : A particle of mass, "m", moves with a constant velocity "v" along a straight line parallel to x-axis as shown in the figure. Find the angular momentum of the particle about the origin of the coordinate system. Also discuss the nature of angular momentum in this case.

Angular momentum of a particle

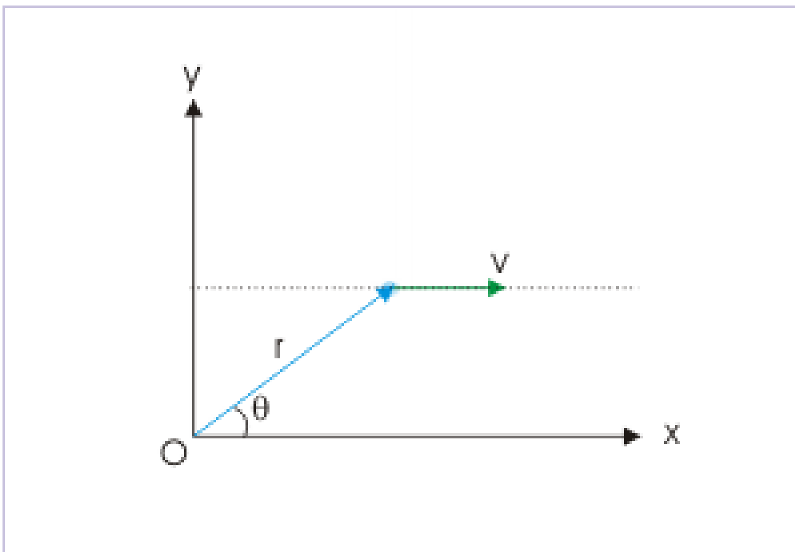

Figure 2: The particle is moving with a constant velocity.

Solution : The magnitude of the angular momentum is given by :

$$\ell = mrv\sin\theta$$

This expression can be rearranged as :

$$\ell = mv\,(r\sin\theta)$$

From the ΔOAC, it is clear that :

Angular momentum of a particle

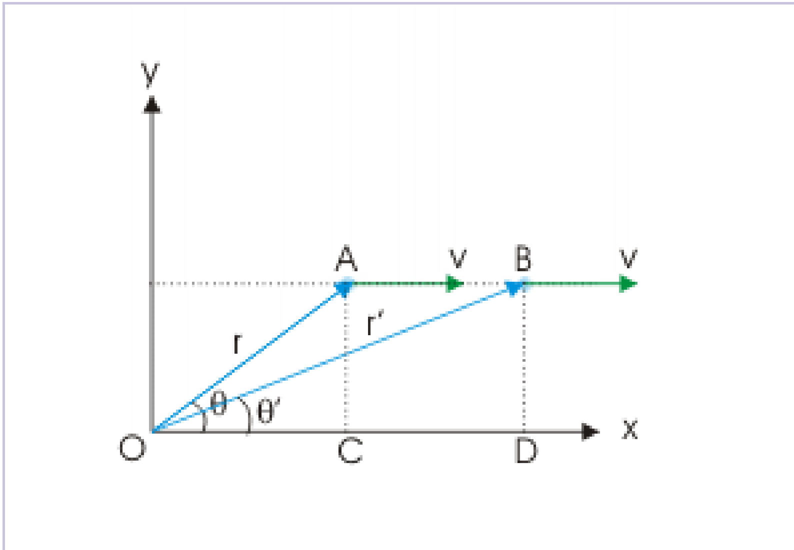

Figure 3: The particle is moving with a constant velocity.

$$r\sin\theta \;=\; \text{AC}$$

At another instant, we have :

$$r'\sin\theta' \;=\; \text{BD}$$

But the perpendicular distance between two parallel lines are same (AC = BD). Thus,

$$\Rightarrow\; r\sin\theta \;=\; \text{a constant}$$

Also, the quantities "m" and "v" are constants. Therefore, angular momentum of the moving particle about origin "O" is a constant.

$$\ell \;=\; mv\;(\;r\sin\theta\;)\;\;=\;\text{a constant}$$

Since angular momentum is constant, its rate of change with time is zero. But, time rate of change of angular momentum is equal to torque (we shall develop this relation in next module). It means that torque on the particle is zero as time derivate of a constant is zero. Indeed it should be so as the particle is not accelerated. This result underlines the fact that the concept of angular momentum is consistent even for the description of linear motion as set out in the beginning of this module.

1.2 Direction of angular momentum

Angular momentum is perpendicular to the plane formed by the pair of position and linear momentum vectors or by the pair of position and velocity vector, depending upon the formula used. Besides, it is also perpendicular to each of operand vectors. However, the vector relation by itself does not tell which side of the plane formed by operands is the direction of torque.

In order to decide the orientation of the angular momentum, we employ right hand vector product rule. The procedure involved is same as that in the case of torque. See the module titled Torque about a point.

1.3 Angular momentum in component form

Angular momentum, being a vector, can be evaluated in component form with the help of unit vectors along the coordinate axes. The various expressions involved in the vector algebraic analysis are as given here :

1: In terms of position and linear momentum vectors

$$\ell = \mathbf{r} \, \mathbf{x} \, \mathbf{p}$$

```
ℓ = | i     j   k |
    | x     y   z |
    | px    py  pz|
```

$$\ell = (y p_z - z p_y) \, \mathbf{i} + (z p_x - x p_z) \, \mathbf{j} + (x p_y - y p_x) \, \mathbf{k} \quad (7)$$

2: In terms of position and velocity vectors

$$\ell = m \, (\mathbf{r} \, \mathbf{x} \, \mathbf{v})$$

```
ℓ = m | i     j     k |
    | x     y    z |
    | vx    vy    vz|
```

$$\ell = m \, (y v_z - z v_y) \, \mathbf{i} + (z v_x - x v_z) \, \mathbf{j} + (x v_y - y v_x) \, \mathbf{k} \quad (8)$$

2 Angular momentum for a particle in rotation

In rotation, a particle rotates about a fixed axis as shown in the figure. We consider haere a particle, which rotates about z-axis along a circular path in a plane parallel to "xy" plane. By the nature of the rotational motion, linear velocity,"**v**", and hence linear momentum, "**p**" are tangential to circular path and are perpendicular to the position vector, "**r**".

A particle rotating about an axis

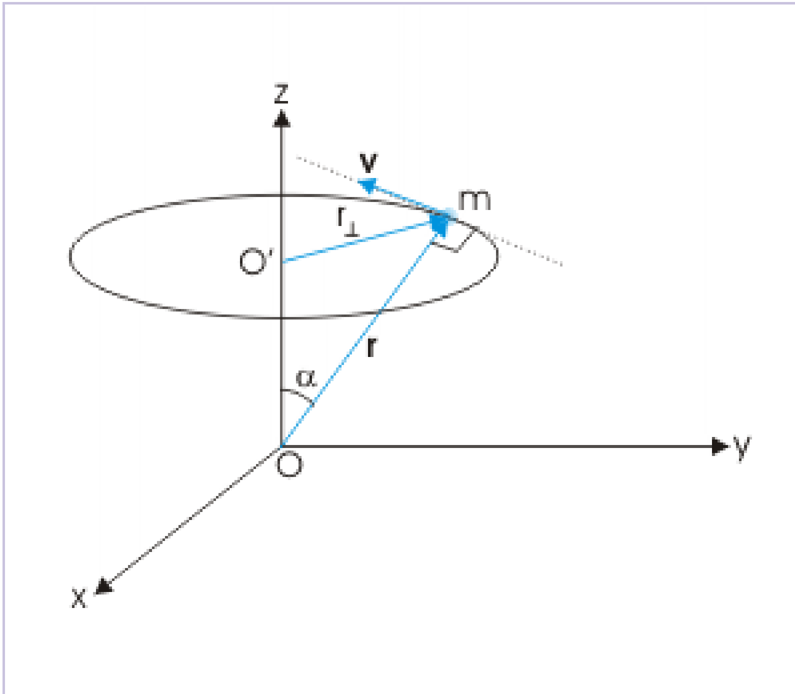

Figure 4: The position vector is perpendicular to velocity vector.

The angle between position vector "**r**" and velocity vector "**v**" is always 90°. There may be some difficulty in visualizing the angle here. In order to visualize the same in a better perspective, we specifically consider a time instant when position vector and moment arm, $r_\perp$, are in "xz" plane. At this instant, the velocity vector, "**v**'", is tangential to the circle and is perpendicular to the "xz" plane. This figure clearly shows that position vector is indeed perpendicular to velocity vector.

Angular momentum

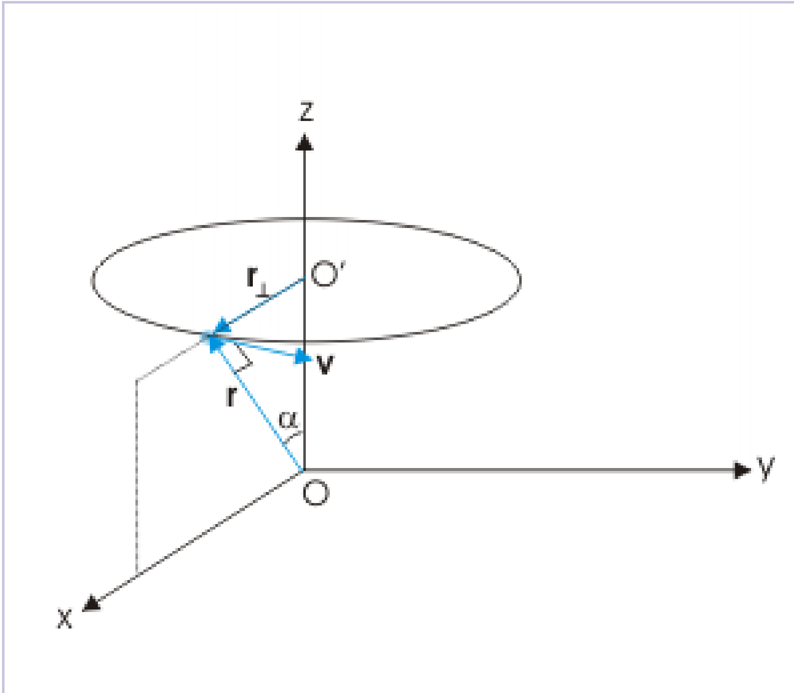

Figure 5: Angular momentum of a particle rotating about an axis.

The angular momentum of the particle, therefore, is :

$$\ell = mrv\sin\theta = mrv\sin 90^0 = mrv$$

The direction of angular momentum is perpendicular to the plane formed by position and velocity vectors. For the specific situation as shown in the figure above, the direction of angular momentum is obtained by first shifting the velocity vector to the origin and then applying right hand rule. Importantly, the angular momentum vector makes an angle with extended x-axis in opposite direction as shown here.

Angular momentum

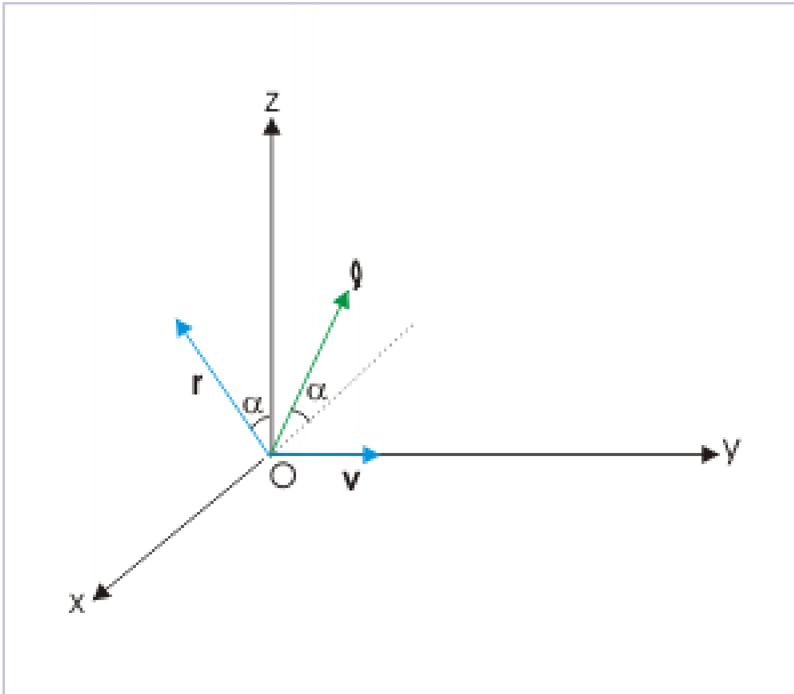

Figure 6: Direction of angular momentum.

We observe that particle is restrained to move along a circular path perpendicular to axis of rotation i.e. z -axis. Thus, only the component of angular momentum in the direction of axis is relevant in the case of rotation. The component of angular momentum in z -direction is :

$$\ell_z = mrv\sin\alpha$$

From geometry, we see that :

$$r\sin\alpha = r_\perp$$

Hence,

$$\ell_z = mr_\perp v = mr_\perp \, x \, \omega r_\perp = mr_\perp{}^2\omega$$

But, we know that $mr_\perp{}^2 = I$. Hence,

$$\ell_z = I\omega$$

This has been the expected relation corresponding to p=mv for translational motion. The product of moment of inertia and angular velocity about a common axis is equal to component of angular momentum about the rotation axis.

If we define angular momentum of the particle for rotation as the product of linear momentum and moment arm about the axis of rotation (not position vector with respect to point "O" as in the general case), then we can say that product of moment of inertia and angular velocity about a common axis is equal to the angular momentum about the rotation axis. Dropping the suffix referring the axis of rotation, we have :

$$\ell = I\omega \tag{9}$$

We must ensure, however, that all quantities in the equation above refer to the same axis of rotation. Also, we should also keep in mind that the definition of angular momentum for rotation about an axis has been equal to the component of linear momentum about the axis of rotation and is different to the one about a point as in the general case. In the nutshell, we find that the angular momentum in rotation is a subset of angular momentum about a point in general motion.

It should be amply clear that the expression of angular momentum in terms of moment of inertia and angular velocity is valid only for rotational motion.

3 Summary

1: Angular momentum of a particle in general motion is given as :

$$\ell = \mathbf{r} \ \mathbf{x} \ \mathbf{p} = m \ (\mathbf{r} \ \mathbf{x} \ \mathbf{v})$$

The interpretation of above relation differs for the reference with respect to which linear distance is measured. In the case of point reference, the vector "r" denotes position vector from the point, whereas it denotes radius vector from the center of circle in rotation.

2: The magnitude of angular momentum is evaluated, using any of the following six relations :

$$\ell = rp\sin\theta$$
$$\ell = rp_{\perp}$$
$$\ell = r_{\perp}p$$
$$\ell = mrv\sin\theta$$
$$\ell = mrv_{\perp}$$
$$\ell = mr_{\perp}v$$

3: The direction of the angular momentum, being a vector product, is evaluated in the same manner as that in the case of torque.

3: In the component form, the angular momentum is expressed as :

```
ℓ = | i    j  k |
    | x    y  z |
    | px   py  pz|
```

and

```
ℓ = m | i    j    k |
      | x    y    z |
      | vx   vy   vz|
```

4: For rotation of a particle, angular momentum has additional expression in terms of moment of inertia and angular velocity as :

$$\ell = I\omega = mr^2\omega$$

where "r " is the radius of the circle from the center lying on the axis of rotation.