Relative motion of projectiles*

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Abstract

Relative motion of two projectiles is a rectilinear motion.

In this module, we shall apply the concept of relative velocity and relative acceleration to the projectile motion. The description here is essentially same as the analysis of relative motion in two dimensions, which was described earlier in the course except that there is emphasis on projectile motion. Besides, we shall extend the concept of relative motion to analyze the possibility of collision between projectiles.

We shall maintain the convention of subscript designation for relative quantities for the sake of continuity. The first letter of the subscript determines the "object", whereas the second letter determines the "other object" with respect to which measurement is carried out. Some expansion of meaning is given here to quickly recapitulate uses of subscripted terms :

$\mathbf{v}_{AB}$: Relative velocity of object "A" with respect to object "B"

v_{ABx} : Component of relative velocity of object "A" with respect to object "B" in x-direction

For two dimensional case, the relative velocity is denoted with bold type vector symbol. We shall , however, favor use of component scalar symbol with appropriate sign to represent velocity vector in two dimensions like in the component direction along the axes of the coordinate system. The generic expression for two dimensional relative velocity are :

In vector notation :

$$\mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B \quad (1)$$

In component scalar form :

$$v_{ABx} = v_{Ax} - v_{Bx} \quad (2)$$

$$v_{ABy} = v_{Ay} - v_{By} \quad (3)$$

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1 Relative velocity of projectiles

The relative velocity of projectiles can be found out, if we have the expressions of velocities of the two projectiles at a given time. Let " $\mathbf{v}_A$ " and " $\mathbf{v}_B$ " denote velocities of two projectiles respectively at a given instant "t". Then :

Relative velocity of projectiles

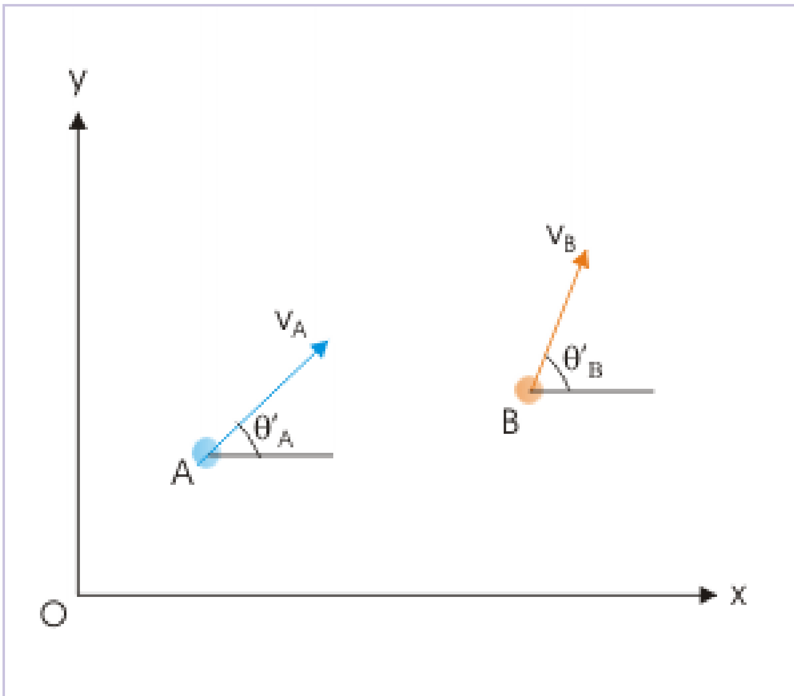

Figure 1: Velocities of projectiles.

$$\mathbf{v}_A = v_{Ax}\mathbf{i} + v_{Ay}\mathbf{j}$$

$$\mathbf{v}_B = v_{Bx}\mathbf{i} + v_{By}\mathbf{j}$$

Hence, relative velocity of projectile "A" with respect to projectile "B" is :

$$\mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B = v_{Ax}\mathbf{i} + v_{Ay}\mathbf{j} - v_{Bx}\mathbf{i} - v_{By}\mathbf{j}$$

$$\mathbf{v}_{AB} = (v_{Ax} - v_{Bx})\,\mathbf{i} + (v_{Ay} - v_{By})\,\mathbf{j} \tag{4}$$

We can interpret this expression of relative velocity as equivalent to consideration of relative velocity in component directions. In the nutshell, it means that we can determine relative velocity in two mutually perpendicular directions and then combine them as vector sum to obtain the resultant relative velocity. Mathematically,

$$\mathbf{v}_{AB} = v_{ABx}\mathbf{i} + v_{ABy}\mathbf{j} \tag{5}$$

where,

$$v_{ABx} = v_{Ax} - v_{Bx}$$

$$v_{ABy} = v_{Ay} - v_{By}$$

This is a significant analysis simplification as study of relative motion in one dimension can be done with scalar representation with appropriate sign.

2 Interpretation of relative velocity of projectiles

The interpretation is best understood in terms of component relative motions. We consider motion in both horizontal and vertical directions.

2.1 Relative velocity in horizontal direction

The interpretation is best understood in terms of component relative motion. In horizontal direction, the motion is uniform for both projectiles. It follows then that relative velocity in horizontal x-direction is also a uniform velocity i.e. motion without acceleration.

Component relative velocity

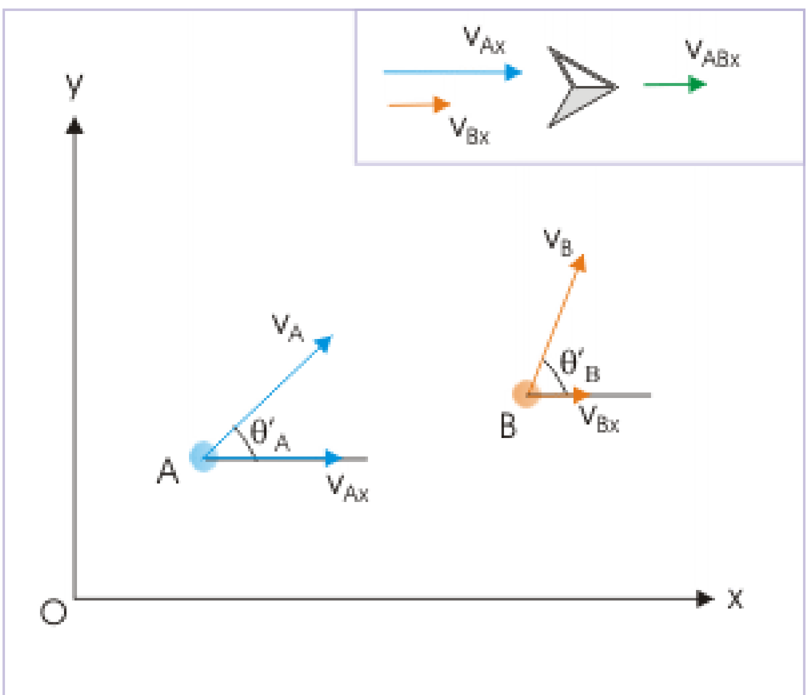

Figure 2: Component relative velocity in x-direction.

The relative velocity in x-direction is :

$$v_{ABx} = v_{Ax} - v_{Bx}$$

As horizontal component of velocity of projectile does not change with time, we can re-write the equation of component relative velocity as :

$$v_{ABx} = u_{Ax} - u_{Bx} \quad (6)$$

Significantly, relative velocity in x-direction is determined by the initial velocities or initial conditions of the projection of two projectiles. It is expected also as components of velocities in x-direction do not change with time. The initial velocity, on the other hand, is fixed for a given projectile motion. As such, horizontal component of relative velocity is constant. A plot of relative velocity in x - direction .vs. time will be a straight line parallel to time axis.

The separation between two objects in x-direction at a given time "t" depends on two factors : (i) the initial separation of two objects in x-direction and (ii) relative velocity in x - direction. The separation in x - direction is given as :

$$\Delta x = x_A - x_B = x_0 + v_{ABx} t$$

where x_0 is the initial separation between two projectiles in x-direction. Clearly, the separation in horizontal direction .vs. time plot would be a straight line with a constant slope. In physical terms, the separation between two projectiles in horizontal direction keeps increasing at a constant rate, which is equal to the magnitude of the component of relative velocity in that direction.

2.2 Relative velocity in vertical direction

The motion in the vertical direction, however, is subject to acceleration due to gravity, which always acts in vertically downward direction. The relative velocity in y-direction is :

Component relative velocity

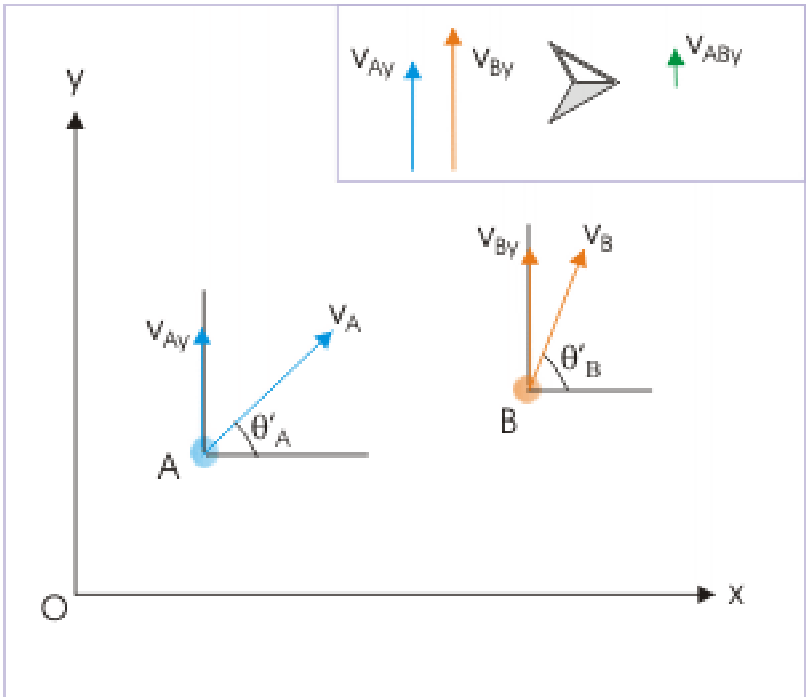

Figure 3: Component relative velocity in y-direction.

$$v_{ABy} = v_{Ay} - v_{By}$$

As vertical component of motion is not a uniform motion, we can use equation of motion to determine velocity at a given time "t" as,

$$v_{Ay} = u_{Ay} - gt$$

$$v_{By} = u_{By} - gt$$

Putting in the expression of relative velocity in y-direction, we have :

$$v_{ABy} = v_{Ay} - v_{By} = u_{Ay} - gt - u_{By} + gt$$

$$\Rightarrow v_{ABy} = u_{Ay} - u_{By} \tag{7}$$

The important aspect of the relative velocity in vertical y-direction is that acceleration due to gravity has not made any difference. The component relative velocity in y-direction is equal to simple difference of components of initial velocities of two projectiles in vertical direction. It is clearly due to the fact the acceleration of two projectiles in y-direction are same i.e. acceleration due to gravity and hence relative acceleration between two projectiles in vertical direction is zero. It means that the nature of relative velocity in vertical direction is same as that in the horizontal direction. A plot of relative velocity in y -direction .vs. time will be a straight line parallel to time axis.

The separation between two objects in y-direction at a given time "t" depends on two factors : (i) the initial separation of two objects in y-direction and (ii) relative velocity in y - direction. The separation in y - direction is given as :

$$\Delta y = y_A - y_B = y_0 + v_{ABy} t$$

where y_0 is the initial separation between two projectiles in y-direction. Note that acceleration term has not appeared in the expression of relative velocity, because they cancel out. Clearly, the separation in vertical direction .vs. time plot would be a straight line with a constant slope. In physical terms, the separation between two projectiles in vertical direction keeps increasing at a constant rate, which is equal to the magnitude of the component of relative velocity in that direction.

3 Resultant relative motion

The component relative velocities in two mutually perpendicular directions have been derived in the previous section as :

Relative velocity of projectiles

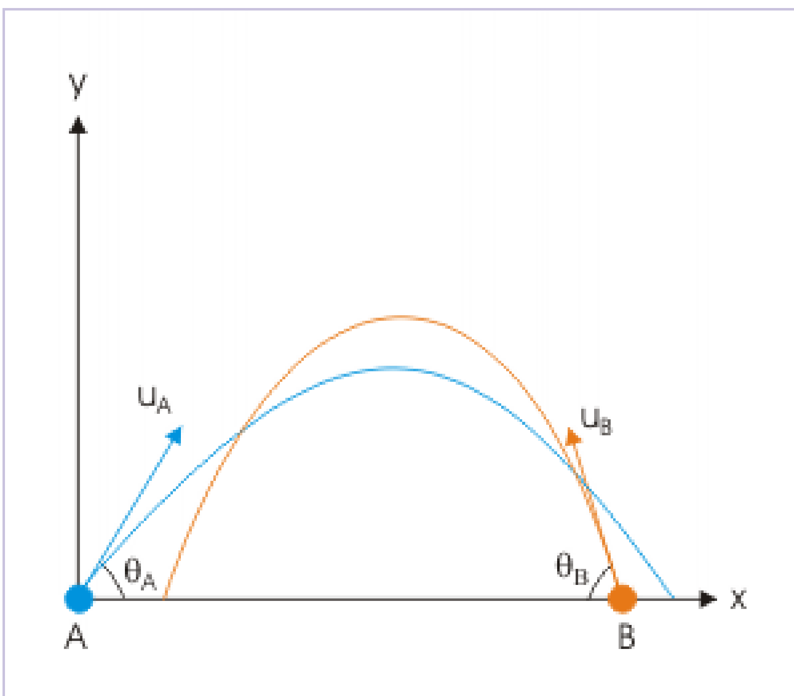

Figure 4: Relative velocity of projectiles depends on initial velocities of projectiles.

$$v_{ABx} = u_{Ax} - u_{Bx}$$

$$v_{ABy} = u_{Ay} - u_{By}$$

These equations are very important results. It means that relative velocity between projectiles is exclusively determined by initial velocities of the two projectiles i.e. by the initial conditions of the two projectiles as shown in the figure below. The component relative velocities do not depend on the subsequent motion i.e. velocities. The resultant relative velocity is vector sum of component relative velocities :

$$\mathbf{v}_{AB} = v_{ABx}\mathbf{i} + v_{ABy}\mathbf{j}$$

Resultant relative velocity of projectiles

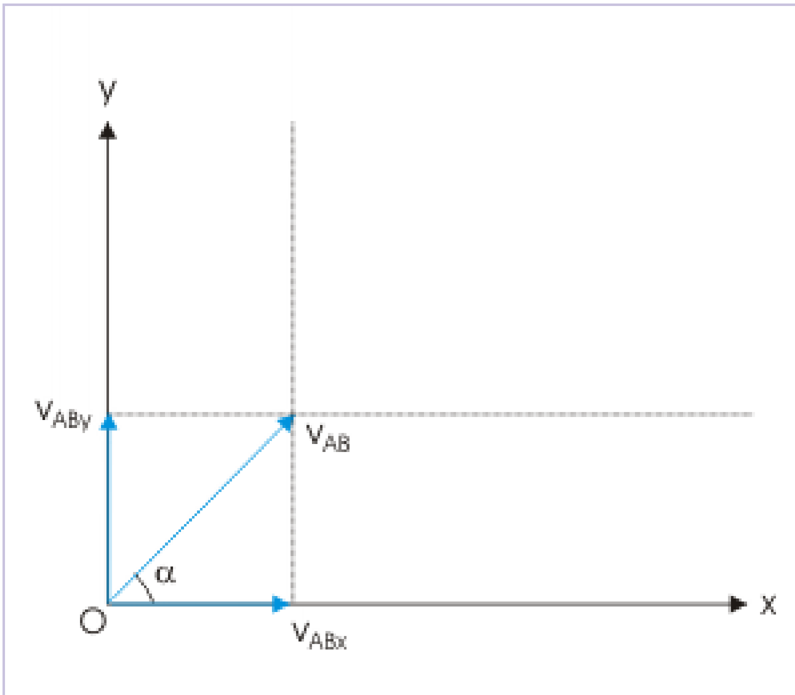

Figure 5: Resultant relative velocity of projectiles is constant.

Since the component relative velocities do not depend on the subsequent motion, the resultant relative velocity also does not depend on the subsequent motion. The magnitude of resultant relative velocity is given by :

$$\Rightarrow v_{AB} = \sqrt{v_{ABx}^2 + v_{ABy}^2} \tag{8}$$

The slope of the relative velocity of "A" with respect to "B" from x-direction is given as :

$$\Rightarrow \tan\alpha = \frac{v_{ABy}}{v_{ABx}} \tag{9}$$

Example 1

Problem : Two projectiles are projected simultaneously from two towers as shown in the figure. Find the magnitude of relative velocity, v_{AB}, and the angle that relative velocity makes with horizontal direction

Relative motion of projectiles

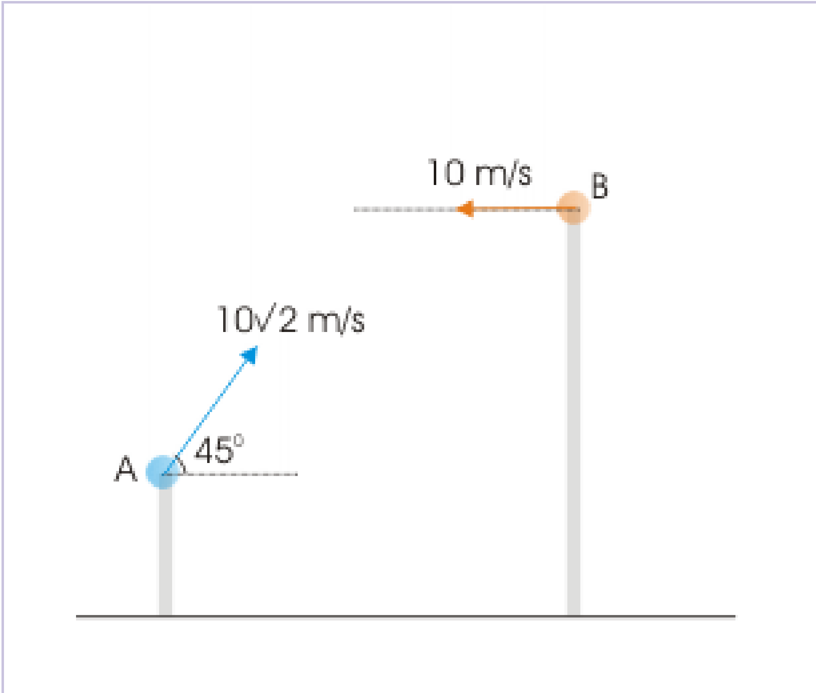

Figure 6: Relative motion of projectiles

Solution : We shall first calculate relative velocity in horizontal and vertical directions and then combine them to find the resultant relative velocity. Let "x" and "y" axes be in horizontal and perpendicular directions. In x-direction,

$$v_{ABx} = u_{Ax} - u_{Bx} = 10\sqrt{2}\cos 45^0 - (-10) = 10\sqrt{2}X\frac{1}{\sqrt{2}} + 10 = 20 \quad m/s$$

In y-direction,

$$v_{ABy} = u_{Ay} - u_{By} = 10\sqrt{2}\cos 45^0 - 0 = 10\sqrt{2}X\frac{1}{\sqrt{2}} = 10 \quad m/s$$

The magnitude of relative velocity is :

$$\Rightarrow v_{AB} = \sqrt{\left(v_{ABx}^2 + v_{ABy}^2\right)} = \sqrt{\left(20^2 + 10^2\right)} = 10\sqrt{5} \quad m/s$$

The angle that relative velocity makes with horizontal is :

$$\Rightarrow \tan\theta = \frac{v_{ABy}}{v_{ABx}} = \frac{10}{20} = \frac{1}{2}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{1}{2}\right)$$

4 Physical interpretation of relative velocity of projectiles

The physical interpretation of the results obtained in the previous section will help us to understand relative motion between two projectiles. We recall that relative velocity can be interpreted by assuming that the reference object is stationary. Consider the expression, for example,

$$v_{ABx} = u_{Ax} - u_{Bx}$$

What it means that relative velocity "vABx" of object "A" with respect to object "B" in x-direction is the velocity of the object "A" in x-direction as seen by the stationary object "B". This interpretation helps us in understanding the nature of relative velocity of projectiles.

Extending the reasoning, we can say that object "A" is moving with uniform motion in "x" and "y" directions as seen by the stationary object "B". The resultant motion of "A", therefore, is along a straight line with a constant slope. This result may be a bit surprising as we might have expected that two projectiles see (if they could) each other moving along some curve - not a straight line.

Relative velocity of projectiles

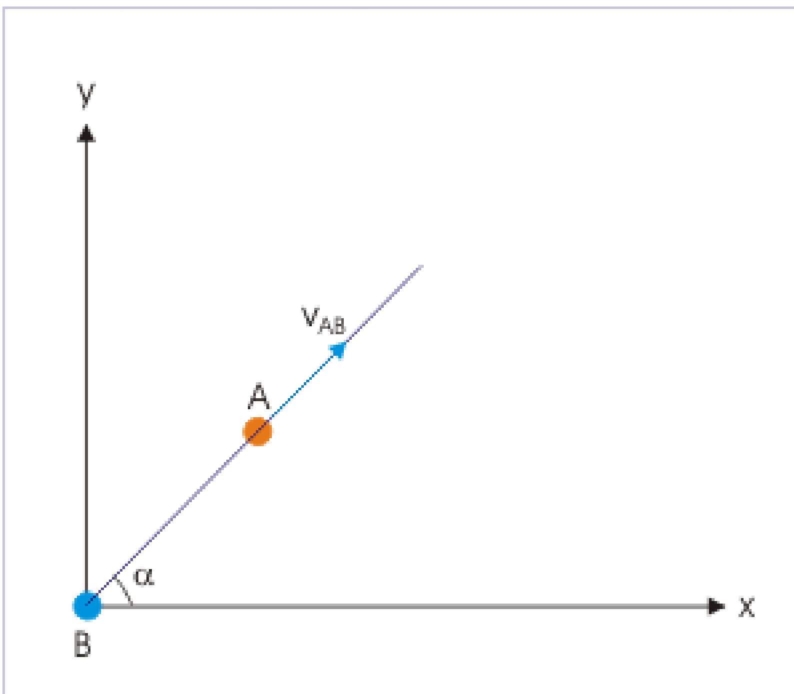

Figure 7: Relative velocity of projectiles is constant.

4.1 Special cases

There are two interesting cases. What if horizontal component of velocities of the two projectiles are same? In this case, relative velocity of projectiles in horizontal direction is zero. Also, it is imperative that there is no change in the initial separation between two projectiles in x-direction. Mathematically,

$$v_{ABx} = u_{Ax} - u_{Bx} = 0$$

There is no relative motion between two projectiles in horizontal direction. They may, however, move with respect to each other in y-direction. The relative velocity in y-direction is given by :

$$v_{ABy} = u_{Ay} - u_{By}$$

Since relative velocity in x-direction is zero, the relative velocity in y-direction is also the net relative velocity between two projectiles.

In the second case, components of velocities in y-direction are equal. In this case, there is no relative velocity in y-direction. The projectiles may, however, have relative velocity in x-direction. As such the relative velocity in y-direction is also the net relative velocity between two projectiles.

5 Exercises

Exercise 1 ***(Solution on p. 11.)***

Two projectiles are projected simultaneously at same speeds, but with different angles of projections from a common point in a given vertical plane. The path of one projectile, as viewed from other projectile is :

(a) a straight line parallel to horizontal
(b) a straight line parallel to vertical
(c) a circle
(d) None of above

Exercise 2 ***(Solution on p. 12.)***

Two projectiles are projected simultaneously at different velocities from a common point in a given vertical plane. If the components of initial velocities of two projectiles in horizontal direction are equal, then the path of one projectile as viewed from other projectile is :

(a) a straight line parallel to horizontal
(b) a straight line parallel to vertical
(c) a parabola
(d) None of above

Solutions to Exercises in this Module

Solution to Exercise 1 (p. 10)

The component relative velocities in horizontal and vertical directions are, defined in terms of initial velocities, which are constant for the given pair of projectiles. Therefore, the relative velocities of two projectiles in horizontal and vertical directions are constants. Let "u_A " and "u_B " be the initial velocities of two projectiles, then component relative velocities in "x" and "y" directions are :

Relative motion of projectiles

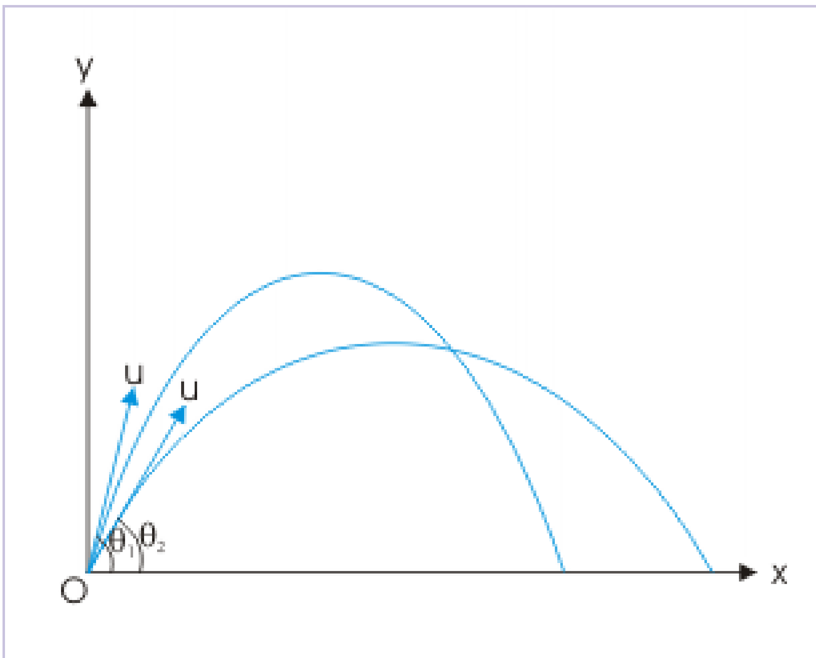

Figure 8: Relative motion of projectiles

$$v_{ABx} = u_{Ax} - u_{Bx}$$

$$v_{ABy} = u_{Ay} - u_{By}$$

The resultant relative velocity is :

$$\mathbf{u}_{AB} = (u_{Ax} - u_{Bx})\,\mathbf{i} + (u_{Ay} - u_{By})\,\mathbf{j}$$

As given in the question, the initial speeds of the projectiles are same, but angles of projections are different. Since sine and cosine of two different angles are different, it follows that component velocities of two projectiles are different in either direction. This is ensured as speeds of two projectiles are same. It implies that components (horizontal or vertical) of the relative velocity are non-zero and finite constant. The resultant relative velocity is, thus, constant, making an angle "θ " with horizontal (x-axis) such that :

Relative motion of projectiles

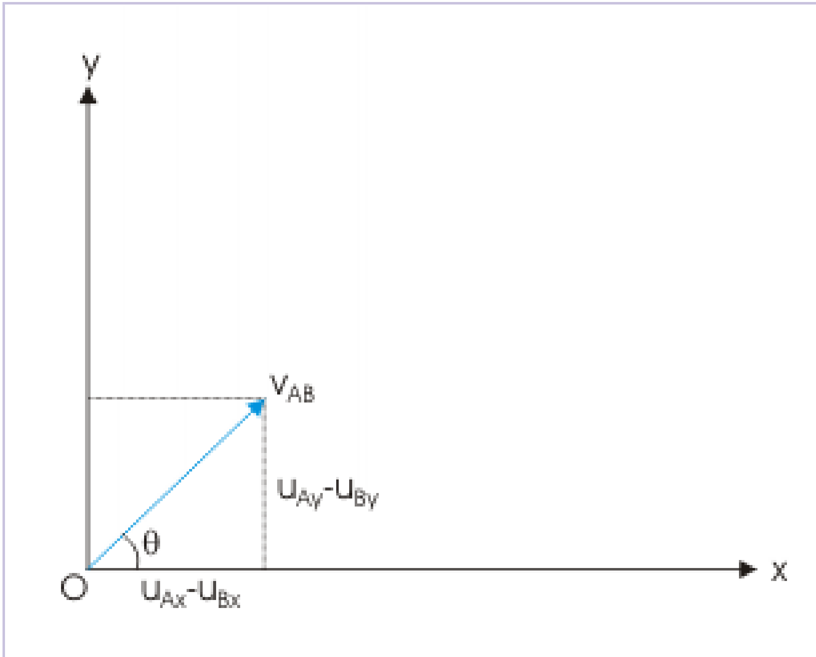

Figure 9: Relative motion of projectiles

$$\tan\theta = \frac{(u_{Ay} - u_{By})}{(u_{Ax} - u_{Bx})}$$

Thus, one projectile (B) sees other projectile (A) moving in a straight line with constant velocity, which makes a constant angle with the horizontal. Hence, option (d) is correct.

Solution to Exercise 2 (p. 10)

The component relative velocities in horizontal and vertical directions are, defined in terms of initial velocities, which are constant for the given projectiles. Let "u_A" and "u_B" be the initial velocities of two projectiles, then component relative velocities in "x" and "y" directions are :

$$v_{ABx} = u_{Ax} - u_{Bx}$$

$$v_{ABy} = u_{Ay} - u_{By}$$

According to the question, components of initial velocities of two projectiles in horizontal direction are equal :

$$u_{Ax} - u_{Bx} = 0$$

It is given that velocities of projections are different. As horizontal components of velocities in horizontal directions are equal, the components of velocities in vertical directions are different. As such, above expression evaluates to a constant vector. Thus, one projectile sees other projectile moving in a straight line parallel to vertical (i.e. direction of unit vector **j**).

Hence, option (b) is correct.