LIMITS*

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Limit is a concept which aims to determine nature of function at a point. This point is infinitesimally close to a declared or test point say "a". We investigate nature of function when independent variable approaches the test value "a" and is not at "a". In the nutshell, we seek to estimate value of function at x=a from a point which is very close to it. Definitely, neither "x" reaches "a" nor f(x) reaches a particular value, say, L. Thus, important thing is to understand that limit denotes correspondence of independent and dependent variables very near but not at the point of estimation.

We should keep in mind while studying limit that it is an estimation based on the behavior of function at points very near to the test point. Limit answers the question : "what would be function value at the test point from its behavior at a point which is very close?". In answering this question, limit considers the nature of function as described by function rule and by estimating value at test point from either direction. This estimate or projection may, however, fail to match actual function value at the test point, if there is a jump or sudden change in function value i.e. when function is discontinuous at the test point. It does not matter. An estimate (limit) remains or exists – if it can be estimated – irrespective of whether it matches function value or not and whether there is a function value at all at the test point or not?

1 Delta – epsilon definition

Idea here is to express nature of function near a point, however, close. We can do this by choosing two very small positive numbers delta (δ) and epsilon ($\in$). We say that limit of function f(x) is L at x = a, if "x" approaches very close to "a", then f(x) approaches very close to L. This means simultaneous closeness :

$$L-\delta < f\left(x\right) < L+\delta \quad \text{for all x in} \quad a-\in < x < a+\in$$

In modulus form :

$$\left|f\left(x\right)-L\right| < \delta \quad \text{for all x in} \quad \left|x-a\right| < \in$$

Limit of function is L, which may or may not be equal to value of function at x=a i.e. f(a). We shall discuss this aspect subsequently.

1.1 Notation

Limit of a function is denoted as :

$$\lim_{x\to a} f\left(x\right) = L$$

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We should read this notation carefully. It is the “limit of function” which is "equal to" L - not the function value. As far as function is concerned it is approaching “L” and value of function, f(a), may or may not be equal to “L”.

1.2 Nature of function

Nature of function is not known by its value at a point. Rather, it is known by the value it is likely to have at a neighboring point. Here, we consider hypothetical set up in order to understand the concept. Our job is to find the approaching value which the function will have and which can be represented as “L”. This we do by determining function value a little to the right towards test point if we approach the test point from left. Similarly, we approach the test point from right by determining function value a little to the left towards test point. If we approach the same value from either side from a very close point, then we say that limit of function at test point is “L”.

Important to note here is that this approaching value of function, L, at the test point, x=a, may or may not be the function value f(a). We should understand that the mechanism of piece-wise function definition allows us to define any function value for any point in the domain of function. Further, if test point is a singularity of domain (a point where function is not defined), then there is no function value at the test point.

1.3 Left hand limit or left limit

Left hand limit is an estimate of function value from a close point on the left of test point. It answer : what would be function value – not what is - at the test point as we approach to it from left? Symbolically, we represent this limit by putting a “minus” sign following test point “a” as “a-“.

$$\lim_{x \to a-} f\left(x\right) = L_l$$

In terms of delta – epsilon definition, we write :

$$L_l - \delta < f\left(x\right) < L_l + \delta \quad \text{for all x in} \quad a- \in < x < a$$

Left hand limit

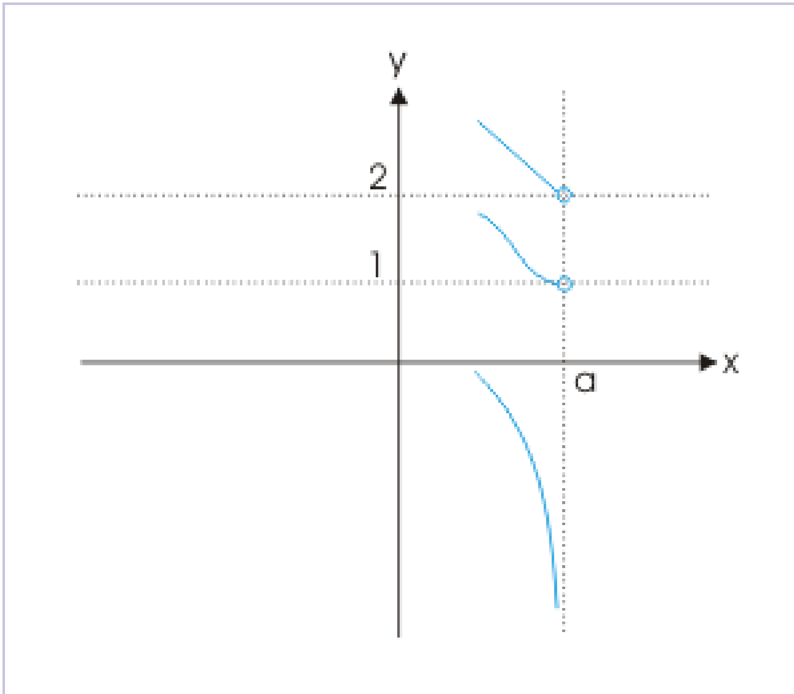

Figure 1: Left hand limit

Graphically, we represent left limit by a curve which points towards limiting value from left terminating with an empty small circle at the test point. The empty circle denotes the limiting value. Since it is an estimate based on nature of graph – not actual function value, it is shown empty. In case, function value is equal to left limit, then circle is filled. If limit approaches infinity, then we show a graph with out terminating circle, approaching an asymptote towards either positive or negative infinity.

1.4 Right hand limit or right limit

Right hand limit is an estimate of function value from a close point on right of test point. It asnwers : what would be function value – not what is - at the test point as we approach to it from right? Symbolically, we represent this limit by putting a “plus” sign following test point “a” as “a+“.

$$\lim_{x \to a+} f\left(x\right) = L_r$$

In terms of delta – epsilon definition, we write :

$$L_l - \delta < f\left(x\right) < L_l + \delta \quad \text{for all x in} \quad a < x < a + \in$$

Right hand limit

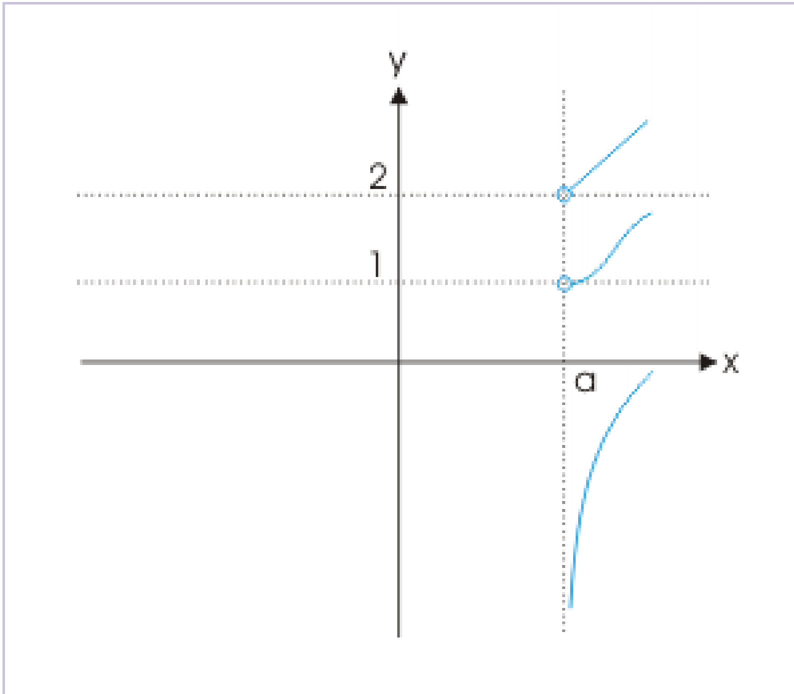

Figure 2: Right hand limit

Graphically, we represent right limit by a curve which points towards limiting value from right terminating with an empty small circle at the test point. If limit approaches infinity, then we show a graph with out terminating circle, approaching an asymptote towards either positive or negative infinity.

1.5 Limit at a point

Limit is an estimate of function value from close points from either side of test point. If left and right limits approach same limiting value, then limit at the point exists and is equal to the common value. Clearly, if left and right limits are not equal, then we can not assign an unique value to the estimate. Clearly, limit of a function answers : what would be function value – not what is - at the test point as we approach to it from either direction? Symbolically, we represent this limit as :

$$\lim_{x \to a} f(x) = L_l = L_r = L$$

In terms of delta – epsilon definition, we write :

$$L - \delta < f(x) < L + \delta \quad \text{for all x in} \quad a - \in < x < a + \in$$

Limit at a point

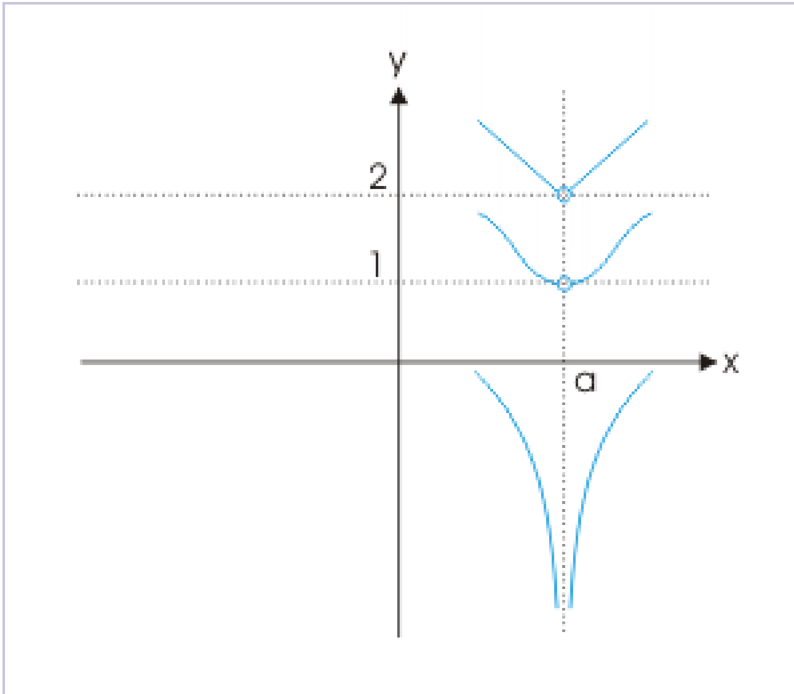

Figure 3: Limit at a point

Graphically, we represent the limit by a pair of curves which point towards limiting value from left and right terminating with a common empty small circle at the test point. If limit approaches infinity, then we show a graph with out terminating circle, approaching an asymptote from either direction in the direction of either positive or negative infinity.

1.6 Limit and continuity

It has been emphasized that limit is an estimate of function value based on function rule at a point. This estimate is not function value. Function value is defined by the definition of function at that point. However, if function is continuous from the neighboring point to the test point, then limit should be equal to function value as well. Consider modulus function :

```
       |  x  ; x>0
f(x) = |  0  ; x=0
       | -x  ; x<0
```

Modulus function

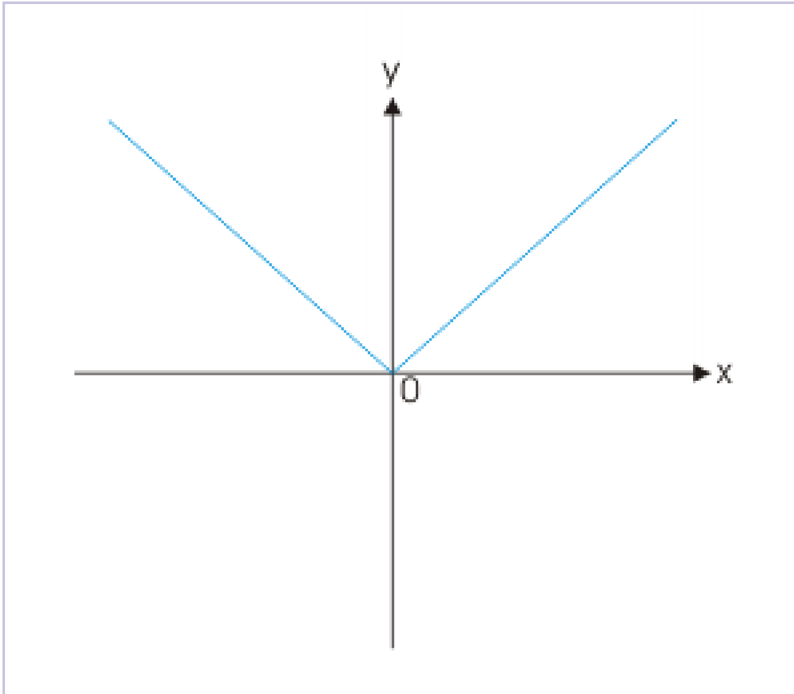

Figure 4: Modulus function

Clearly, at test point x=0,

$$L = f(0) = 0$$

This is an important result which gives us a method to determine limit of a function. If function is continuous, simply put the test value into function definition. The value of function is limit of function at that point.

1.7 Limit and discontinuity

If function rule changes exactly at the test point, then limit of the function, L, and value of function, f(a), are not same. In order to clearly understand the implication of the statement about inequality of limit and function value, we consider a modification to the modulus function :

```
        |  x;  x>0
f(x) =  |  1;  x=0
        | -x;  x<0
```

Modified modulus function

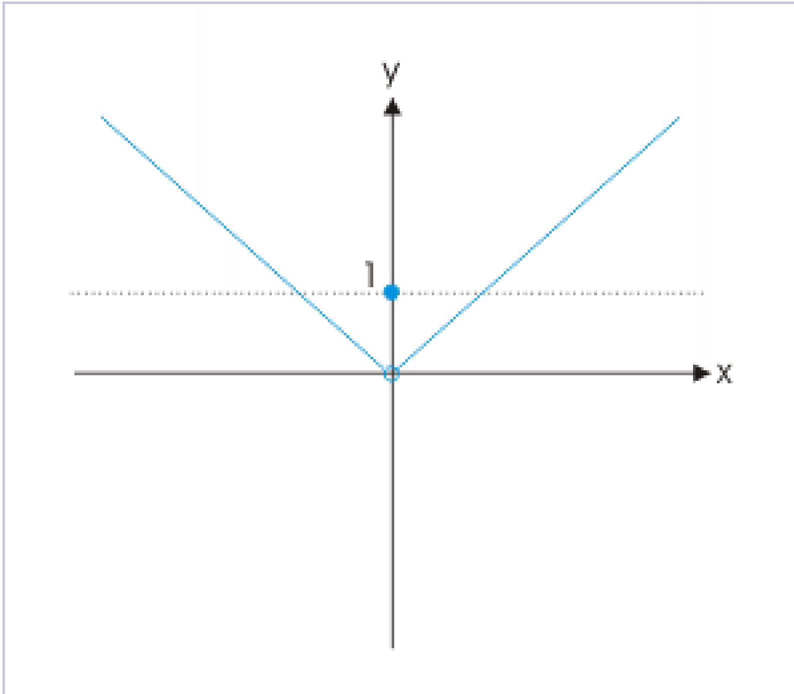

Figure 5: Modified modulus function

At test point x=0,

$$L = 0$$

$$f(0) = 1$$

$$L \neq f(0)$$

Therefore, limit of function exists at x=0 even though it is not equal to function value. This is an important result which gives us a method to determine limit of a piece-wise defined functions. We need to evaluate function from both left and right side. If limits are equal from both sides, then limit of function at test point is equal to either limit. However, if left and right limits are not equal then limit of function does not exist at the test point.

1.8 Limit and singularity

Singularity or exception point is a point where function is not defined. It is outside definition of function. However, function can point (or tend or approach) to a value at a point where it is not defined. Limit as we know estimate value from a close point where function exits and can project a value based on function definition at points very close to exception point. Consider limit of a rational function :

$$\lim_{x \to 1} \frac{(x-1)(x+3)}{(x-1)}$$

Rational function

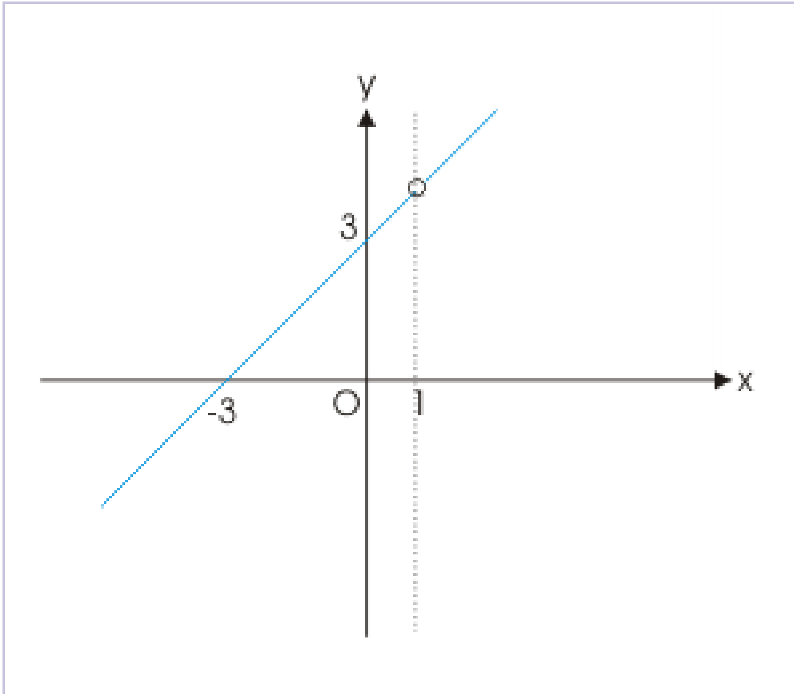

Figure 6: Rational function

The singularity of function is obtained by setting denominator to zero. Thus, singularity exists at x=1. We want to know nature of function about this point. In other words, we want to know what would have been the value of function at this point had the function been defined there. For this, we need to evaluate left and right limit at this point. Graphically, there is a hole in the graph of the function. How can we estimate value of function at a point if it is not defined there ?

We keep linear factor in the denominator to know singularity. Extrapolating value at the singularity is a reverse process. We need to calculate function value in the neighborhood, where function is defined. For this, we require to remove linear factor from the denominator. Canceling out (x-1) from both numerator and denominator, we have :

$$\lim_{x \to 1} \frac{(x-1)(x+3)}{(x-1)} = \lim_{x \to 1} (x+3) = 4$$

Thus, function is not defined at x=1, but its limit at the point is 4. This means that nature of function in its immediate vicinity is such that the function should have attained a value of 4 had it been estimated on the basis of nature of function in the neighborhood.

This is again an important result which gives us a method to determine limit of a function, when function is not defined at certain point or has other indeterminate or meaningless forms. We need to simplify function expression till we get a form which can be evaluated.

Let us now consider reciprocal function :

$$f(x) = \frac{1}{x}$$

Its singularity is obtained by setting denominator to zero. Thus, singularity exists x=0 for the reciprocal function. As such, domain of function is R-{0}. In order to know the function, we need to know nature of function in the vicinity of undefined point. We can do this by evaluating limit on either side of the singularity.

Reciprocal function

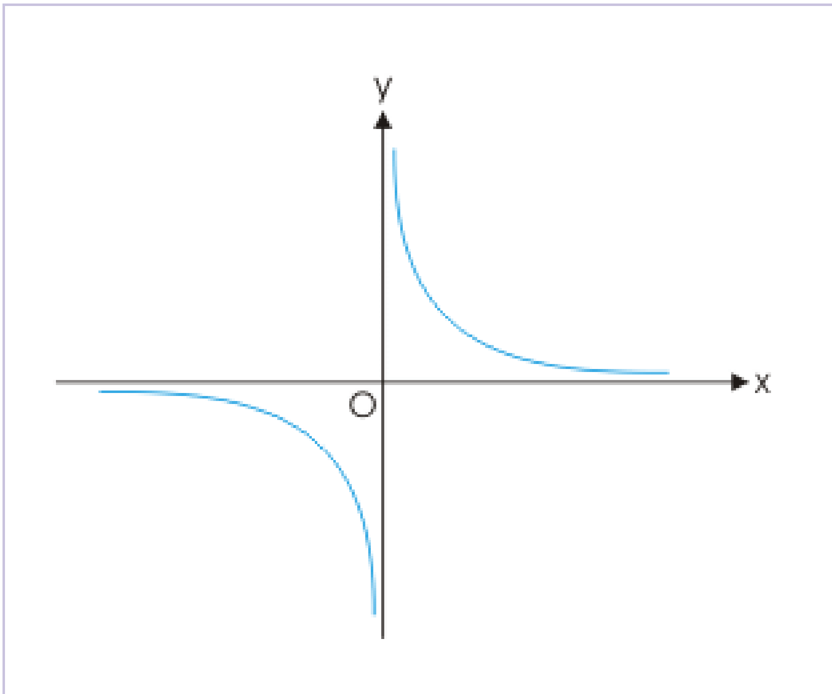

Figure 7: Reciprocal function

$$\lim_{x \to 0-} \frac{1}{x} = -\infty$$

$$\lim_{x \to 0+} \frac{1}{x} = \infty$$

Important point to underscore here is that limiting values of x or f(x) as infinity is a valid estimates. To be more explicit, value of function can approach infinity as limiting value. In this case, left and right limits are not same. Therefore, limit of function does not exist at exception point x=0. In order to explore limit at exception point, we consider the case of modulus of reciprocal function. In this case also, function is not defined at x=0. But, for x<0; |x| = -x and for x>0; |x| = x. Hence, left and right limits are :

$$\lim_{x \to 0-} |\frac{1}{x}| = \lim_{x \to 0-} -\frac{1}{x} = \infty$$

$$\lim_{x \to 0+} |\frac{1}{x}| = \lim_{x \to 0+} \frac{1}{x} = \infty$$

Modulus of reciprocal function

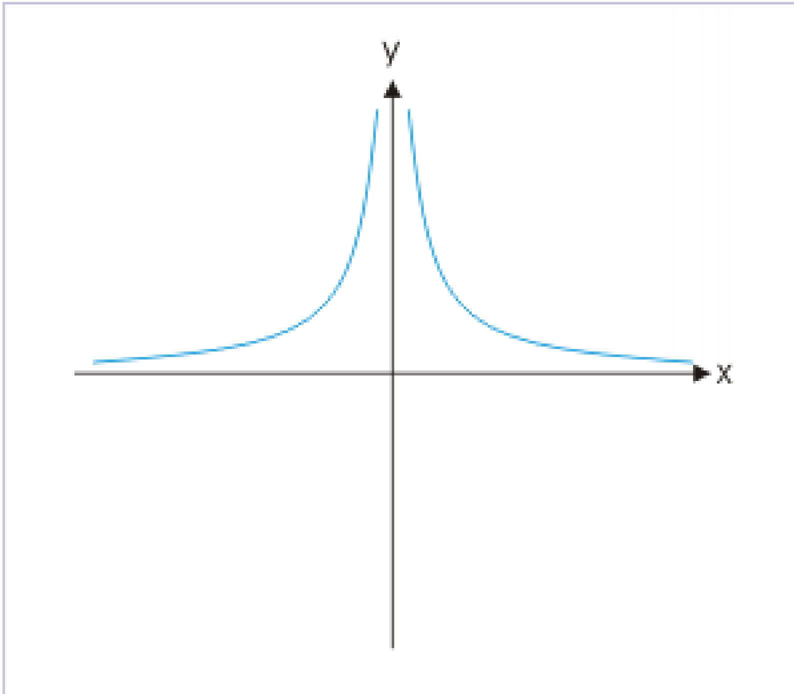

Figure 8: Modulus of reciprocal function

In this case, left and right limits are equal. Therefore, limit of function exist at exception point x=0 and it is given as :

$$\lim_{x \to 0} \left|\frac{1}{x}\right| = \infty$$

2 Limits and infinity

We have noted that limit of function can be positive or negative infinity to reflect the estimate that function value is expected to be either very large positive or negative value. It happens when a finite value is divided by a value which is exceedingly small. If the divisor is a exceedingly small negative value, then function approaches negative infinity and if the divisor is a exceedingly small positive value, then function approaches positive infinity. Similar intuitive limiting values involving infinity are given here :

(1) Let "a" be a finite real number.

$$\lim_{x \to \infty} \frac{a}{x} = 0$$

$$\lim_{x \to -\infty} \frac{a}{x} = 0$$

(2) Let "a" be a finite real number.

$$\lim_{x \to 0-} \frac{a}{x} = -\infty$$

$$\lim_{x \to 0+} \frac{a}{x} = \infty$$

(3) Let "a" be a finite real number.

$$\lim_{x \to \infty} ax = \infty; \quad a > 0$$

$$= 0; \quad a = 0$$

$$= -\infty; \quad a < 0$$

(4) Let "a" be a finite real number.

$$\lim_{x \to \infty} a^x = \infty; \quad a > 1$$

$$= 1; \quad a > 0$$

$$= -\infty; \quad 0 < a < 1$$

3 Indeterminate limit forms

The indeterminate limit form is also called meaningless form. There are seven such forms in total. We, however, need to be careful in interpreting these forms. The interpretation is most important part of evaluation of limit. For example, if we say that 0/0 is indeterminate limit form, then we mean that both numerator and denominator of function approach zero, but none are equal to zero. In the example below, both numerator and denominator approach to zero as x approaches 2 :

$$\lim_{x \to 2} \frac{(x^2 - 4)}{(x - 2)}$$

As x approaches 2, both numerator and denominator approaches to zero. Therefore, the function expression is an indeterminate form 0/0. However, following is not an indeterminate form :

$$\lim_{x \to 0} \frac{0}{x} = \infty$$

In the limit given above, the numerator is zero (not approaches to zero), whereas denominator approaches to zero. Thus, the rational form is determinate form and approaches infinity and the limit is also infinity.

In addition to 0/0 indeterminate form, there are other indeterminate forms which needs to be converted to determinate form so that limit can be evaluated. The seven indeterminate forms are :

$$0/0, \quad \infty/\infty, \quad 0.\infty, \quad \infty - \infty, \quad 0^{0,} \quad \infty^{0,} \quad 1^{\infty}$$

Again, we emphasize to distinguish interpretation with each of the indeterminate limit forms given above. In order to clarify the point further, let us consider two limits :

$$\lim_{x \to \pi/2} \sin x^{\tan x}$$

$$\lim_{x \to \pi/2} 1^{\tan x} = 1$$

In the first case, the base is approaching 1 and exponent is approaching infinity. Hence, it is indeterminate limit form. In the second case, base is 1 – not approaching to 1. Hence, it is not an indeterminate form and is evaluated to 1.

4 Evaluation of limit

There are three distinct regimes based on the discussion as above. We evaluate limit in accordance with following algorithm :

1: The function is not in indeterminate function form.

We simply plug in the value of test point into the function. The function value is equal to the limit of function.

2: The function is in indeterminate function form.

We transform the functions into determinate form. There are many techniques to transform an indeterminate form. Rationalization, simplification etc are important means to change forms. Expansion series of transcendental functions are also helpful. Besides, there are forms whose limits are known. We attempt to structure given expression in those standard forms and then find the limit. Finally, there are specific algorithms depending on nature of function, which help to remove indeterminate form and find limit. We shall study specific techniques in specific context.

3: The function is piecewise defined.

Using two approaches outlined above, we determine left and right limit and see whether they are equal or not? If equal, then limit is equal to either of left and right limits.

Note : We shall divide evaluation of limit in separate categories for different function types. The evaluations of limits shall be dealt separately in detail. Here, we work with few fundamental limits only.

5 Example

Example 1

Problem : Plot the graph of function and determine limit when x->1.

$$f\left(x\right)=\frac{x+3}{x-1}$$

Solution : Here, singularity exists at x=1. The function is not defined at this point. Rearranging, we have :

$$\frac{x+3}{x-1}=\frac{x-1+4}{x-1}=1+\frac{4}{x-1}$$

Here core graph is 4/x graph. We obtain graph of 4/(x-1) by shifting graph of 4/x towards right by 1 unit. In order to obtain the graph of given function, we shift the graph of 4/(x-1) up by 1 unit. We see that the function has one zero at x=-3. It has one asymptote at x=1. On the other hand, the y-intercept of function is -3.

Graph of rational function

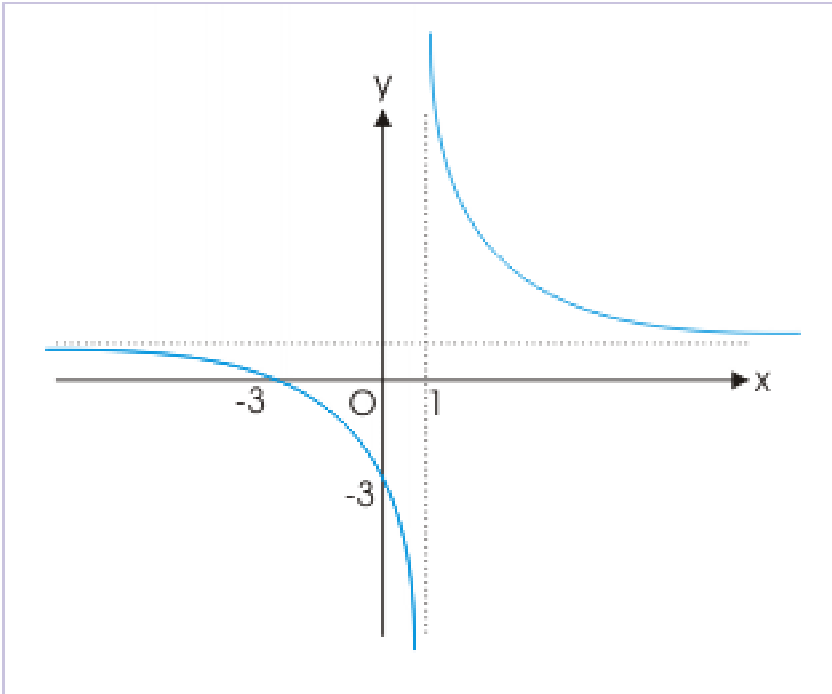

Figure 9: Graph of rational function

Left hand limit is :

$$\Rightarrow \lim_{x \to 1-} 1 + \frac{4}{x-1} = -\infty$$

Right hand limit is :

$$\Rightarrow \lim_{x \to 1+} 1 + \frac{4}{x-1} = \infty$$

Since left and right hand limits are not equal, the limit of function does not exist as x approaches to 1.

6 Exercises

Exercise 1 ***(Solution on p. 15.)***

Determine limit :

$$\Rightarrow \lim_{n \to \infty} \frac{n}{n+1}$$

Exercise 2 ***(Solution on p. 15.)***

Determine limit :

$$\lim_{x\to 0} \cos x$$

Exercise 3 *(Solution on p. 15.)*

Determine limit :

$$\lim_{x\to 0} \log_{0.5} x$$

Exercise 4 *(Solution on p. 15.)*

Determine limit :

$$\lim_{x\to 0} \frac{1}{x-1}$$

Exercise 5 *(Solution on p. 15.)*

Determine limit :

$$\lim_{x\to \infty} \frac{x^2}{x^2}$$

Exercise 6 *(Solution on p. 15.)*

Determine limit :

$$\lim_{x\to 0} \frac{|x^3|}{x}$$

Exercise 7 *(Solution on p. 15.)*

Determine limit :

$$\lim_{x\to 1} \frac{\left(x^2-1\right)}{|x-1|}$$

Solutions to Exercises in this Module

Solution to Exercise 1 (p. 13)

The limit has ∞/∞ indeterminate form. Dividing each term of numerator and denominator by n,

$$\Rightarrow \frac{n}{n+1} = \frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}} = \frac{1}{1+\frac{1}{n}}$$

As n→∞, 1/n→0. Hence,

$$\Rightarrow \lim_{n\to\infty} \quad \frac{n}{n+1} = 1$$

Solution to Exercise 2 (p. 13)

The limit has determinate form. Since cosx is a continuous function, its limit is equal to its value at x=0 i.e. cos0 = 1. Hence,

$$\Rightarrow \lim_{x\to 0} \quad \cos x = 1$$

Solution to Exercise 3 (p. 14)

The limit has determinate form. Here, $\log_{0.5}x$ is a continuous function. The base of exponential function is less than 1. As x approaches zero, the function approaches positive infinity (refer its graph).

$$\Rightarrow \lim_{x\to 0} \quad \log_{0.5}x = \infty$$

Solution to Exercise 4 (p. 14)

Note that we are not testing limit at singularity. The function is determinate form. Hence, limit is :

$$\Rightarrow \lim_{x\to 0} \quad \frac{1}{x-1} = \frac{1}{0-1} = -1$$

Solution to Exercise 5 (p. 14)

The limit has ∞/∞ indeterminate form. Simplifying,

$$\Rightarrow \lim_{x\to\infty} \quad 1 = 1$$

Solution to Exercise 6 (p. 14)

The limit has 0/0 indeterminate form. For $x < 0$, $|x^3| = -x^3$. Hence,

$$\Rightarrow \frac{|x^3|}{x} = -\frac{x^3}{x} = -x^2$$

$$\Rightarrow \lim_{x\to 0-} \quad -x^2 = 0$$

For $x > 0$, $|x^3| = x^3$. Hence,

$$\Rightarrow \frac{|x^3|}{x} = \frac{x^3}{x} = x^2$$

$$\Rightarrow \lim_{x\to 0+} \quad x^2 = 0$$

Clearly, $L_l = L_r = L$. Thus,

$$\Rightarrow \lim_{x\to 0} \quad \frac{|x^3|}{x} = 0$$

Solution to Exercise 7 (p. 14)

The limit has 0/0 indeterminate form. For $x < 1, \quad |x-1| = -(x-1)$. Hence,

$$\frac{(x^2-1)}{|x-1|} = \frac{(x^2-1)}{-(x-1)} = -(x-1)$$

$$\Rightarrow \lim_{x \to 1} \quad \frac{(x^2-1)}{|x-1|} = \lim_{x \to 1} \quad -(x+1) = -2$$

For $x > 1, \quad |x-1| = (x-1)$. Hence,

$$\frac{(x^2-1)}{|x-1|} = \frac{(x^2-1)}{(x-1)} = (x-1)$$

$$\Rightarrow \lim_{x \to 1+} \quad \frac{(x^2-1)}{|x-1|} = \lim_{x \to 1+} \quad (x+1) = 2$$

Clearly, $L_l \neq L_r$. Hence, given limit does not exists.