Gravitational field due to rigid bodies*

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Abstract

A rigid body is an aggregation of small elements, which can be treated as point mass.

1 Gravitational field of rigid bodies

We shall develop few relations here for the gravitational field strength of bodies of particular geometric shape without any reference to Earth's gravitation.

Newton's law of gravitation is stated strictly in terms of point mass. The expression of gravitational field due to a particle, as derived from this law, serves as starting point for developing expressions of field strength due to rigid bodies. The derivation for field strength for geometric shapes in this module, therefore, is based on developing technique to treat a real body mass as aggregation of small elements and combine individual effects. There is a bit of visualization required as we need to combine vectors, having directional property.

Along these derivations for gravitational field strength, we shall also establish Newton's shell theory, which has been the important basic consideration for treating spherical mass as point mass.

The celestial bodies - whose gravitational field is appreciable and whose motions are subject of great interest - are usually spherical. Our prime interest, therefore, is to derive expression for field strength of solid sphere. Conceptually, a solid sphere can be considered being composed of infinite numbers of closely packed spherical shells. In turn, a spherical shell can be conceptualized to be aggregation of thin circular rings of different diameters.

The process of finding the net effect of these elements fits perfectly well with integration process. Our major task, therefore, is to suitably set up an integral expression for elemental mass and then integrate the elemental integral between appropriate limits. It is clear from the discussion here that we need to begin the process in the sequence starting from ring –> spherical shell –> solid sphere.

1.1 Gravitational field due to a uniform circular ring

We need to find gravitational field at a point "P" lying on the central axis of the ring of mass "M" and radius "a". The arrangement is shown in the figure. We consider a small mass "dm" on the circular ring. The gravitational field due to this elemental mass is along PA. Its magnitude is given by :

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Gravitational field due to a ring

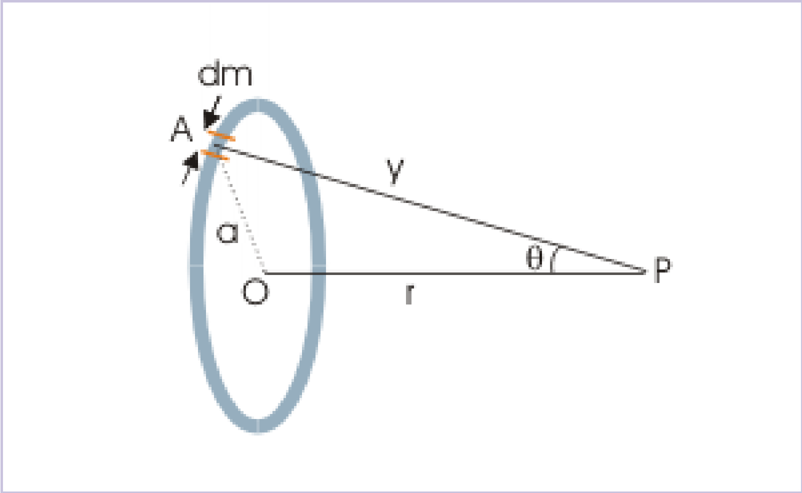

Figure 1: The gravitational field is measured on axial point "P".

$$dE = \frac{Gdm}{PA^2} = \frac{Gdm}{(a^2 + r^2)}$$

We resolve this gravitational field in the direction parallel and perpendicular to the axis in the plane of OAP.

Gravitational field due to a ring

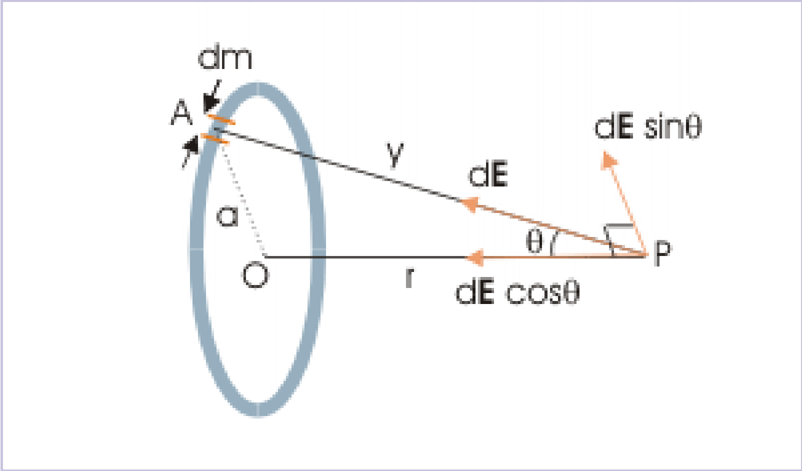

Figure 2: The net gravitational field is axial.

$$dE_{||} = dE\cos\theta$$

$$dE_{\perp} = dE\sin\theta$$

We note two important things. First, we can see from the figure that measures of "y" and "θ" are same for all elemental mass. Further, we are considering equal elemental masses. Therefore, the magnitude of gravitational field due any of the elements of mass "dm" is same, because they are equidistant from point "P".

Second, perpendicular components of elemental field intensity for pair of elemental masses on diametrically opposite sides of the ring are oppositely directed. On integration, these perpendicular components will add up to zero for the whole of ring. It is clear that we can assume zero field strength perpendicular to axial line, if mass distribution on the ring is uniform. For uniform ring, the net gravitational intensity will be obtained by integrating axial components of elemental field strength only. Hence,

Gravitational field due to a ring

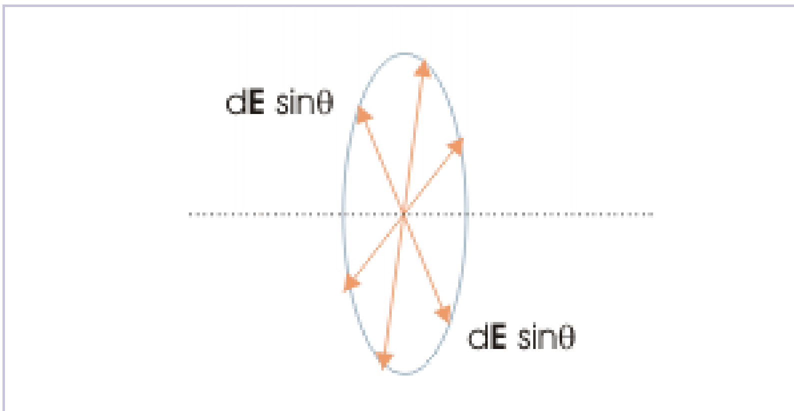

Figure 3: Perpendicular components cancel each other.

$$\Rightarrow E = \int dE\cos\theta$$

$$\Rightarrow E = \int \frac{Gdm\cos\theta}{(a^2 + r^2)}$$

The trigonometric ratio "cosθ" is a constant for all points on the ring. Taking out cosine ratio and other constants from the integral,

$$\Rightarrow E = \frac{G\cos\theta}{(a^2 + r^2)} \int dm$$

Integrating for m = 0 to m = M, we have :

$$\Rightarrow E = \frac{GM\cos\theta}{(a^2 + r^2)}$$

From triangle OAP,

$$\Rightarrow \cos\theta = \frac{r}{(a^2 + r^2)^{\frac{1}{2}}}$$

Substituting for "cosθ" in the equation ,

$$\Rightarrow E = \frac{GMr}{\left(a^2 + r^2\right)^{\frac{3}{2}}}$$

For r = 0, E = 0. The gravitation field at the center of ring is zero. This result is expected also as gravitational fields due to two diametrically opposite equal elemental mass are equal and opposite and hence balances each other.

1.1.1 Position of maximum gravitational field

We can get the maximum value of gravitational field by differentiating its expression w.r.t linear distance and equating the same to zero,

$$\frac{dE}{dr} = 0$$

This yields,

$$\Rightarrow r = \frac{a}{\sqrt{2}}$$

Substituting in the expression of gravitational field, the maximum field strength due to a circular ring is :

$$\Rightarrow E_{\text{max}} = \frac{GMa}{2^{\frac{1}{2}}\left(a^2 + \frac{a^2}{2}\right)^{\frac{3}{2}}} = \frac{GMa}{3\sqrt{3}a^2}$$

The plot of gravitational field with axial distance shows the variation in the magnitude,

Gravitational field due to a ring

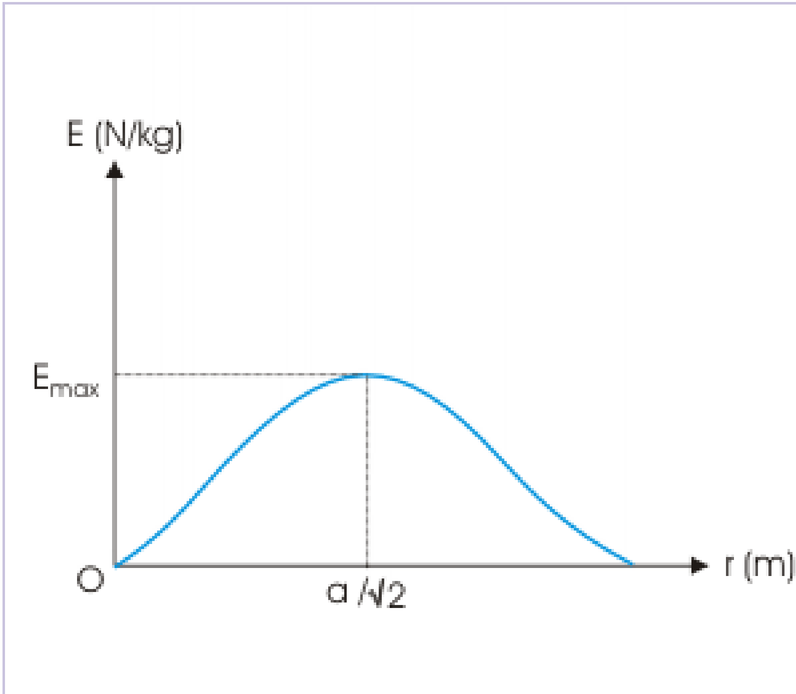

Figure 4: The gravitational field along the axial line.

1.2 Gravitational field due to thin spherical shell

The spherical shell of radius "a" and mass "M" can be considered to be composed of infinite numbers of thin rings. We consider one such ring of infinitesimally small thickness "dx" as shown in the figure. We derive the required expression following the sequence of steps as outlined here :

Gravitational field due to thin spherical shell

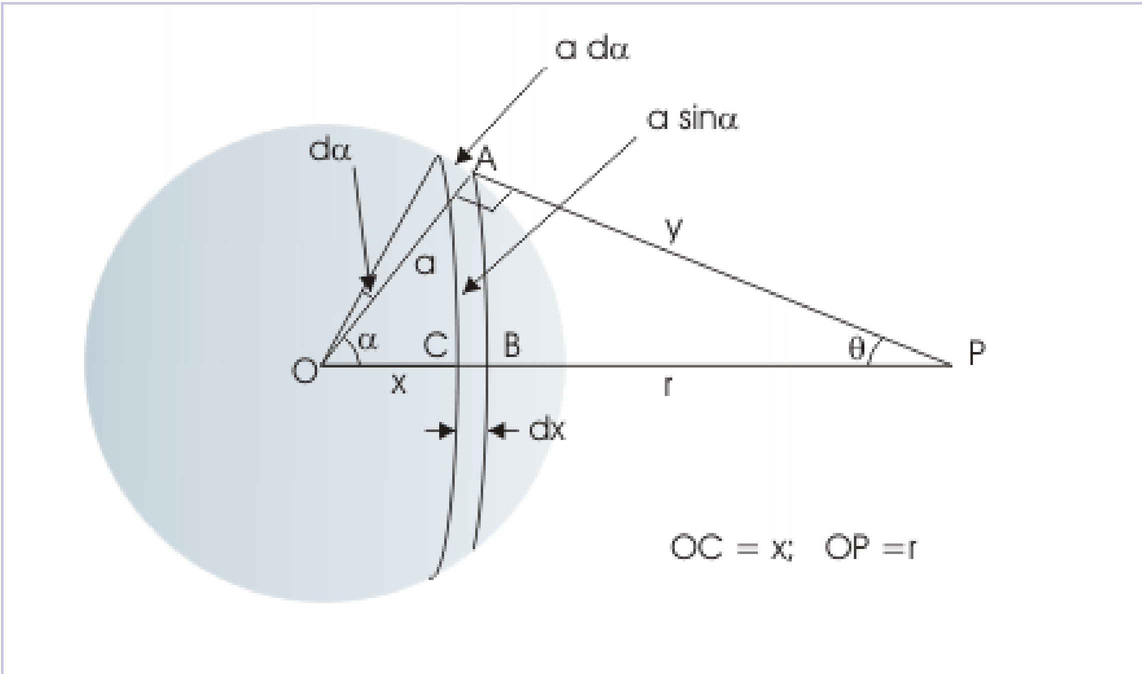

Figure 5: The gravitational field is measured on axial point "P".

(i) Determine mass of the elemental ring in terms of the mass of shell and its surface area.

$$dm = \frac{M}{4\pi a^2} X 2\pi a \text{sin}\alpha dx = \frac{M a \text{sin}\alpha dx}{2a^2}$$

From the figure, we see that :

$$dx = a d\alpha$$

Putting these expressions,

$$\Rightarrow dm = \frac{M a \text{sin}\alpha dx}{2a^2} = \frac{M a \text{sin}\alpha a d\alpha}{2a^2} = \frac{M \text{sin}\alpha d\alpha}{2}$$

(ii) Write expression for the gravitational field due to the elemental ring. For this, we employ the formulation derived earlier for the ring,

$$\Rightarrow dE = \frac{G dm \text{cos}\theta}{AP^2}$$

Putting expression for elemental mass,

$$\Rightarrow dE = \frac{G M \text{sin}\alpha d\alpha \text{cos}\theta}{2y^2}$$

(v) Set up integral for the whole disc

We see here that gravitational fields due to all concentric rings are directed towards the center of spherical shell along the axis.

$$\Rightarrow E = GM\int \frac{\sin\alpha\cos\theta d\alpha}{2y^2}$$

The integral expression has three varibles "α","θ" and "y".Clearly, we need to express variables in one variable "x". From triangle, OAP,

$$\Rightarrow y^2 = a^2 + r^2 - 2ar\cos\alpha$$

Differentiating each side of the equation,

$$\Rightarrow 2ydy = 2ar\sin\alpha d\alpha$$

$$\Rightarrow \sin\alpha d\alpha = \frac{ydy}{ar}$$

Again from triangle OAP,

$$\Rightarrow a^2 = y^2 + r^2 - 2yr\cos\theta$$

$$\Rightarrow \cos\theta = \frac{y^2 + r^2 - a^2}{2yr}$$

Putting these values in the integral,

$$\Rightarrow E = GM\int \frac{dy\left(y^2 + r^2 - a^2\right)}{4ar^2y^2}$$

$$\Rightarrow E = GM\int \frac{dy}{4ar^2}\left(1 - \frac{a^2 - r^2}{y^2}\right)$$

We shall decide limits of integration on the basis of the position of point "P" – whether it lies inside or outside the shell. Integrating expression on right side between two general limits, initial (L_1) and final (L_2),

$$\Rightarrow E = GM\int_{L_1}^{L_2} \frac{dy}{4ar^2}\left(1 - \frac{a^2 - r^2}{y^2}\right)$$

$$\Rightarrow E = \frac{GM}{4ar^2}\left[y + \frac{a^2 - r^2}{y}\right]_{L1}^{L2}$$

1.2.1 Evaluation of integral for the whole shell

Case 1 : The point "P" lies outside the shell. The total gravitational field is obtained by integrating the integral from y = r-a to y = r+a,

$$\Rightarrow E = \frac{GM}{4ar^2}\left[y + \frac{a^2 - r^2}{y}\right]_{r-a}^{r+a}$$

$$\Rightarrow E = \frac{GM}{4ar^2}\left[r + a + \frac{a^2 - r^2}{r + a} - r + a - \frac{a^2 - r^2}{r - a}\right]$$

$$\Rightarrow E = \frac{GM}{4ar^2}\left[2a + \left(a^2 - r^2\right)\left(\frac{1}{r + a} - \frac{1}{r - a}\right)\right]$$

$$\Rightarrow E = \frac{GM}{4ar^2} X4a$$

$$\Rightarrow E = \frac{GM}{r^2}$$

This is an important result. We have been using this result by the name of Newton's shell theory. According to this theory, a spherical shell, for a particle outside it, behaves as if all its mass is concentrated at its center. This is how we could calculate gravitational attraction between Earth and an apple. Note that radius of the shell, "a", does not come into picture.

Case 2 : The point "P" lies outside the shell. The total gravitational field is obtained by integrating the integral from x = a-r to x = a+r,

$$\Rightarrow E = \frac{GM}{4ar^2} \left[y + \frac{a^2 - r^2}{y} \right]_{a-r}^{a+r}$$

$$\Rightarrow E = \frac{GM}{4ar^2} \left[a + r + \frac{a^2 - r^2}{a + r} - a + r - \frac{a^2 - r^2}{a - r} \right]$$

$$\Rightarrow E = \frac{GM}{4ar^2} \left[2r + \left(a^2 - r^2 \right) \left(\frac{1}{a + r} - \frac{1}{a - r} \right) \right]$$

$$\Rightarrow E = \frac{GM}{4ar^2} \left[2r - 2r \right] = 0$$

This is yet another important result, which has been used to determine gravitational acceleration below the surface of Earth. The mass residing outside the sphere drawn to include the point below Earth's surface, does not contribute to gravitational force at that point.

The mass outside the sphere is considered to be composed of infinite numbers of thin shells. The point within the Earth lies inside these larger shells. As gravitational intensity is zero within a shell, the outer shells do not contribute to the gravitational force on the particle at that point.

A plot, showing the gravitational field strength, is shown here for regions both inside and outside spherical shell :

Gravitational field due to thin spherical shell

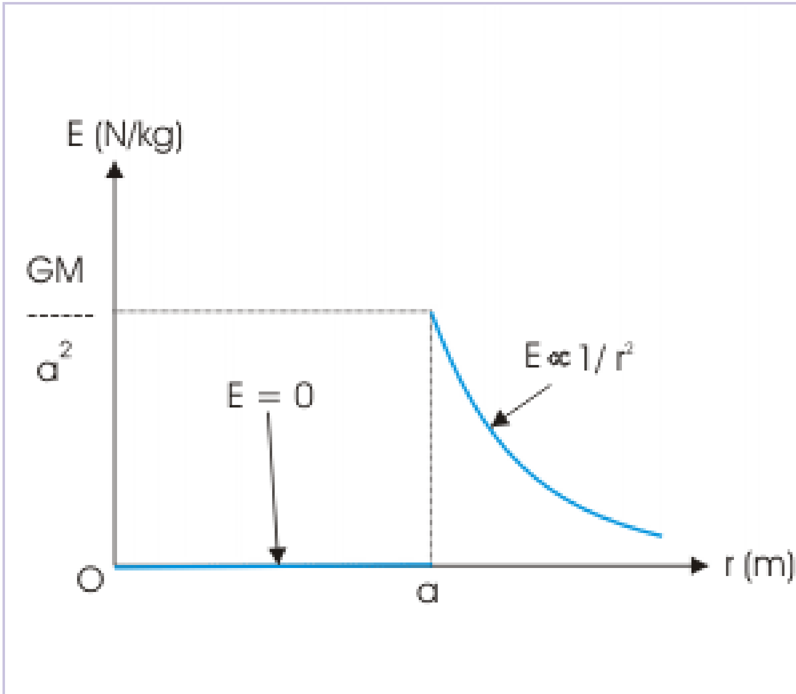

Figure 6: The gravitational field along linear distance from center.

1.3 Gravitational field due to uniform solid sphere

The uniform solid sphere of radius "a" and mass "M" can be considered to be composed of infinite numbers of thin spherical shells. We consider one such spherical shell of infinitesimally small thickness "dx" as shown in the figure. The gravitational field strength due to thin spherical shell at a point outside shell, which is at a linear distance "r" from the center, is given by

Gravitational field due to solid sphere

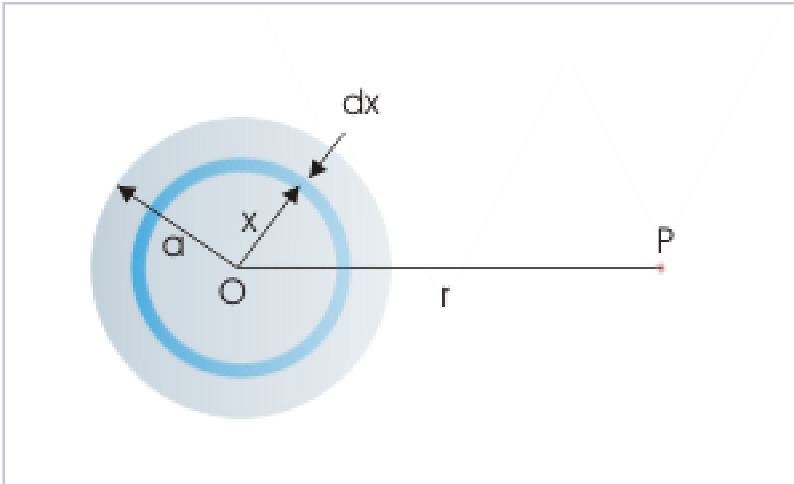

Figure 7: The gravitational field at a distance "r" from the center of sphere.

$$dE = \frac{Gdm}{r^2}$$

The gravitational field strength acts along the line towards the center of sphere. As such, we can add gravitational field strengths of individual shells to obtain the field strength of the sphere. In this case, most striking point is that the centers of all spherical shells are coincident at one point. This means that linear distance between centers of spherical shell and the point ob observation is same for all shells. In turn, we can conclude that the term " r^2 " is constant for all spherical shells and as such can be taken out of the integral,

$$\Rightarrow E = \int \frac{Gdm}{r^2} = \frac{G}{r^2} \int dm = \frac{GM}{r^2}$$

We can see here that a uniform solid sphere behaves similar to a shell. For a point outside, it behaves as if all its mass is concentrated at its center. Note that radius of the sphere, "a", does not come into picture. Sphere behaves as a point mass for a point outside.

1.3.1 Gravitational field at an inside point

We have already derived this relation in the case of Earth.

For this reason, we will not derive this relation here. Nevertheless, it would be intuitive to interpret the result obtained for the acceleration (field strength) earlier,

Gravitational field inside solid sphere

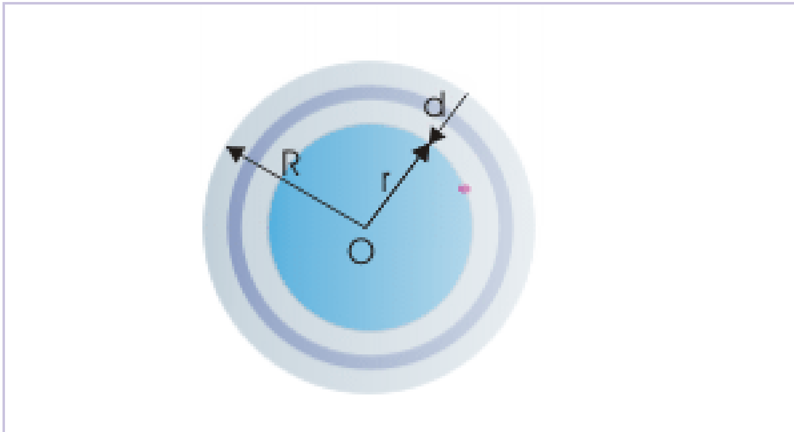

Figure 8: The gravitational field at a distance "r" from the center of sphere.

$$\Rightarrow g\prime = g_0\left(1 - \frac{d}{R}\right)$$

Putting value of "g0" and simplifying,

$$\Rightarrow g\prime = \frac{GM}{R^2}\left(1 - \frac{d}{R}\right) = \frac{GM}{R^2}\left(\frac{R-d}{R}\right) = \frac{GMr}{R^3}$$

As we have considered "a" as the radius of sphere here – not "R" as in the case of Earth, we have the general expression for the field strength insider a uniform solid sphere as :

$$\Rightarrow E = \frac{GMr}{a^3}$$

The field strength of uniform solid sphere within it decreases linearly within "r" and becomes zero as we reach at the center of the sphere. A plot, showing the gravitational field strength, is shown here for regions both inside and outside :

Gravitational field due to uniform solid sphere

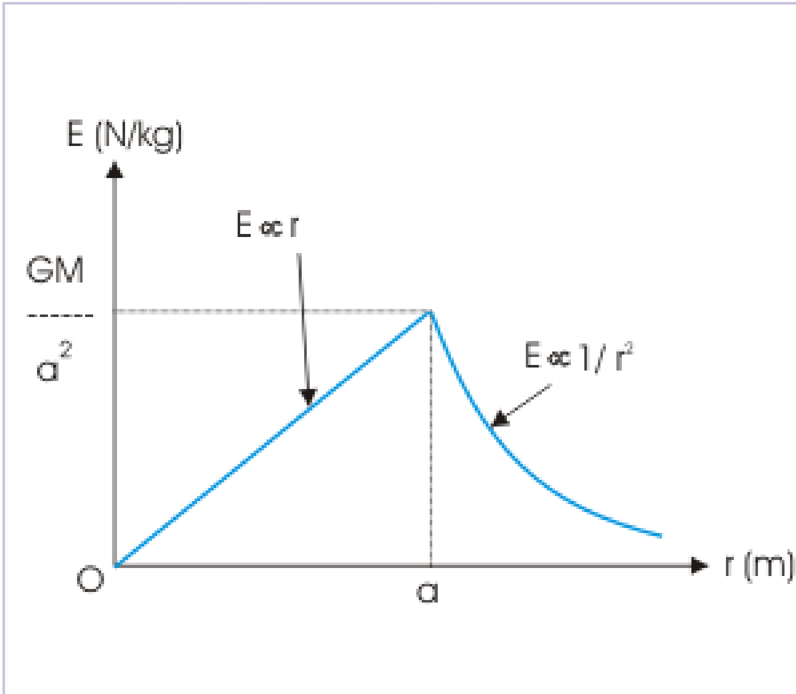

Figure 9: The gravitational field along linear distance from center.