Subsets*

Sunil Kumar Singh

The collections are generally linked in a given context. If we think of ourselves, then we belong to a certain society, which in turn belongs to a province, which in turn belongs to a country and so on. In the context of a school, all students of a school belong to school. Some of them belong to a certain class. If there are sections within a class, then some of these belong to a section.

We need to have a mathematical relationship between different collections of similar types. In set theory, we denote this relationship with the concept of "subset".

Definition 1: Subset

A set, "A" is a subset of set "B", if each member of set "A" is also a member of set "B".

We use symbol " $\subset$ " to denote this relationship between a "subset" and a "set". Hence,

$$A \subset B$$

We read this symbolic representation as : set "A" is a subset of set "B". We express the intent of relationship as :

$$A \subset B \quad \text{if} \quad x \in A, \quad \text{then} \quad x \in B$$

It is evident that set "B" is larger of the two sets. This is sometimes emphasized by calling set "B" as the "superset" of "A". We use the symbol " $\supset$ " to denote this relation :

$$B \supset A$$

If set "A" is not a subset of "B", then we write this symbolically as :

$$A \not\subset B$$

1 Important results / deductions

Some of the important characteristics and related deductions are presented here :

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1.1 Equality of two sets

It is clear that set "B" is inclusive of subset "A". It means that "B" may have additional elements over and above those common with "B". In case, all elements of "B" are also in "A", then two sets are equal. We express this symbolically as :

$$If \quad A \subset B \quad \text{and} \quad B \subset A, \quad \text{then} \quad A = B.$$

This is true in other direction as well :

$$If \quad A = B, \quad \text{then} \quad A \subset B \quad \text{and} \quad B \subset A.$$

We can write two instances in a single representation as :

$$A \subset B \quad \text{and} \quad B \subset A \Leftrightarrow A = B$$

The symbol " $\Leftrightarrow$ " means that relation holds in either direction.

1.2 Relation with itself

Every set is subset of itself. This is so because every element is present in itself.

1.3 Relation with Empty set

Empty set is a subset of every set. This deduction is direct consequence of the fact that empty set has no element. As such, this set is subset of all sets.

2 Proper subset

We have seen from the deductions above that special circumstance of "equality" can blur the distinction between "set" and "subset". In order to emphasize, mother-child relation between sets, we coin the term "proper subset". If every element of set "B" is not present in set "A", then "A" is a "proper" subset of set "B"; otherwise not. This means that set "B" is a larger set, which, besides other elements, also includes all elements of set "A".

Set of vowels in English alphabet, "V", is a "proper" subset of set of English alphabet, "E". All elements of "V" are present in "E", but not all elements of "E" are present in "V".

There is a bit of conventional differences. Some write a "proper" subset relation using symbol " $\subset$ " and write symbol " $\subseteq$ " to mean possibility of equality as well. We have chosen not to differentiate two subset types.

3 Number system

The number system is one such system, in which different number groups are related. Natural number is a subset of integers. integers are subset of rational numbers and rational numbers are subset of real numbers. None of these sets are equal. Hence, relations are described by proper subsets.

$$N \subset Z$$

We can write the chain of relation among number sets :

$$\Rightarrow P \subset N \subset Z \subset Q \subset R$$

However, irrational numbers are also subset of real numbers, but irrational numbers is not rational numbers. We represent this relation by emphasizing that rational numbers is not a subset of irrational numbers or vice-versa. We depict this relation as :

$$Q\,(\text{rational numbers}) \not\subset T\,(\text{irrational numbers})$$

But irrational numbers is subset of real numbers. The real numbers comprises of only two subsets at the highest level – rational and irrational. Therefore, irrational numbers is the remaining collection after deducting rational numbers from real numbers.

Following the logic, we define set of irrational numbers as :

$$T\,(\text{irrational numbers}) = \{x : x \in R \quad \text{and} \quad x \notin Q\}$$

4 Power set

Power set is formed of all possible subsets of a given set. It is denoted as P(A).

Definition 2: Power set
The collection of all subsets of a set "A" is called power set, P(A).

For example, consider a set given by :

$$A = \{1, 3, 4\}$$

What are the possible subsets? There are three subsets consisting of individual elements: {1}, {3} and {4}. Then, elements taken two at a time form following subsets : {1,3}, {1,4} and {3,4}. Since order or sequence does not matter in set representation, there are only three subsets of two elements taken together. Now, the elements taken three at a time form the only one subset : {1,3,4}. Remember, a set is a subset of itself. Further, empty set (ϕ) is subset of any set. Hence, ϕ is also a subset of the given set "A".

The set comprising of all possible subsets of given set "A" is :

$$P\,(A) = \{\phi, \{1\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{3, 4\}, \{1, 3, 4\}\}$$

We note two important points from this representation of power set :

1: The elements of a power set are themselves sets. In other words, every element of a power set is a set.

2: If the numbers of elements (cardinality) in a set is "n", then numbers of elements in power set is 2^n .

For a set having three elements, the total numbers of elements in the power set is :

$$\Rightarrow m = 2^n = 2^3 = 8$$

We can see that this result is consistent with the illustration given above. We should, here, emphasize to avoid confusion that counting of elements of a set (cardinality) excludes empty set. It is, however, counted as members of power set.

5 Example

Problem 1: The finite sets "A" and "B" have "m" and "n" numbers of elements respectively. The total numbers of subsets of "A" is 56 more than the total numbers of subsets of "B". Find "m" and "n".

Solution : According to relation obtained for power set, the total numbers of subsets of "A" and "B" are :

$$k_A = 2^m$$

$$k_B = 2^n$$

According to question,

$$k_A - K_B = 56$$

$$\Rightarrow 2^m - 2^n = 56$$

We need to find two equations to find "m" and "n". For this we seek expansion of "56" in terms of powers of "2".

$$56 = 8X7 = 8\,(8-1) = 2^3\left(2^3 - 1\right)$$

In order to get this form, we rearrange the expression on the LHS of the earlier equation as :

$$\Rightarrow 2^n\left(2^{m-n} - 1\right) = 2^3\left(2^3 - 1\right)$$

Equating powers of similar base,

$$n = 3 \quad \text{and} \quad m = 6$$

6 Intervals

Intervals is an alternative way to represent a subset of real numbers. Real numbers is represented by a number line having infinite membership. We can think any segment of this number line as subset or interval. Consider an interval, where "a" and "b" belongs to real numbers and a $<$ b :

$$a < x < b$$

The value of "x" falls between "a" and "b". For example, an interval 2<x<4 is a collection of all points lying between end points 2 and 4. The important thing is that this interval does not include end points and is called "open" interval. We can represent this collection as a set in "set builder form" as :

$$\{x : x \in R \quad and \quad 2 < x < 4\}$$

Alternatively, we can use pair of small brackets to represent open interval as :

$$(2, 4)$$

The two forms of representations are equivalent. The later form is obviously an easier and convenient representation of the subset of real number. We use small braket "(" or ")" to denote interval that excludes end point. Likewise, we use square bracket "[" or "]" to denote interval that includes end point. We can represent a "close" interval as [2,4]. This interval is equivalent to :

$$[2, 4] = 2 \le x \le 4$$

We can have combination of "open" and "close" brackets like :

$$(2, 4] = 2 < x \le 4$$

As a reminder, we should note that interval corresponding to real numbers or its subset is an infinite set as we can have infinite points on the line segment corresponding to an interval.

6.1 Graphical representation

The graphical representation uses a segment of line on the number line representing real numbers. The line segment is demarcated by a pair of two small circles – a filled circle to mean that end point is included in the interval and an unfilled circle to mean that end point is excluded from the interval.

Let us consider a,b ∈ R and a < b, then

$$(a,b) = a < x < b$$

$$[a,b) = a \le x < b$$

$$(a,b] = a < x \le b$$

$$[a,b] = a \le x \le b$$

Graphically,

Intervals

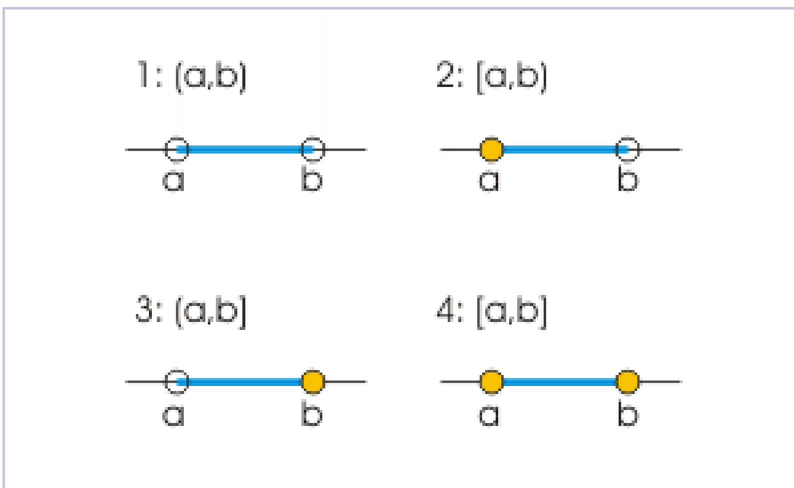

Figure 1: Representation on real number line.

6.2 Set of real numbers

The real numbers is represented graphically by a straight line. The question that we seek to be answer here is whether the set of integers is bounded by infinity. In other words, whether we can define interval of real numbers like :

$$[-\infty, \infty]$$

The literal meaning of infinity is "unboundedness". Infinity is considered as a large number, which may either be positive or negative. It does not have a finite (fixed) value. Infinity, therefore, is not a part of real number system. It does not lie on the real number line. For this reason, we can not assign infinity to a real variable like (though we do generally):

$$x = \infty$$

It follows, then, that appropriate interval, representing real numbers, is open at both ends :

$$R = -\infty < x < \infty = (-\infty, \infty)$$

6.2.1 Interval of real numbers greater than or less than a given value

In the interval form, we can write the set of real numbers greater than a given value, "a", as :

$$a < x < \infty = (a, \infty)$$

This is equivalent to :

$$x > a$$

The final notation "x > a" does not require to mention about infinity. It is an interval of real numbers greater than the given value 'a' appearing on the right. It is implied that it can be any large value. Similarly, the interval of real numbers less than a given value is :

$$-\infty < x < a = (-\infty, a)$$

$$x < a$$