Intersection of sets*

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We have pointed out that a set representing a real situation is not an isolated collection. Sets, in general, overlaps with each other. It is primarily because a set is defined on few characteristics, whereas elements generally can possess many characteristics. Unlike union, which includes all elements from two sets, the intersection between two sets includes only common elements.

Definition 1: Intersection of two sets
The intersection of sets "A" and "B" is the set of all elements common to both "A" and "B".

The use of word "and" between two sets in defining an intersection is quite significant. Compare it with the definition of union. We used the word "or" between two sets. Pondering on these two words, while deciding membership of union or intersection, is helpful in application situation.

The intersection operation is denoted by the symbol, " $\cap$ ". We can write intersection in set builder form as :

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Intersection of two sets

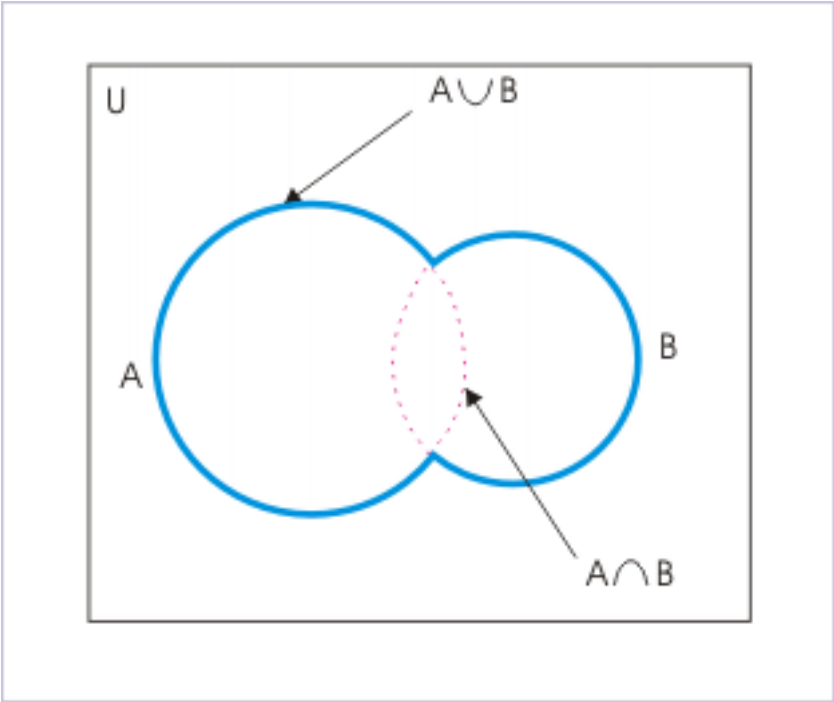

Figure 1: The intersection set consists of elements common to two sets.

$$A \cap B = \{x : \quad x \in A \quad and \quad x \in B\}$$

Again note use of the word "and" in set builder qualification. We can read this as "x" is an element, which belongs to set "A" and set "B". Hence, it means that "x" belongs to both "A" and "B".

In order to understand the operation, let us consider the earlier example again,

$$A = \{1, 2, 3, 4, 5, 6\}$$

$$B = \{4, 5, 6, 7, 8\}$$

Then,

$$A \cap B = \{4, 5, 6\}$$

On Venn diagram, an intersection is the region intersected by circles, which represent two sets.

Intersection of two sets

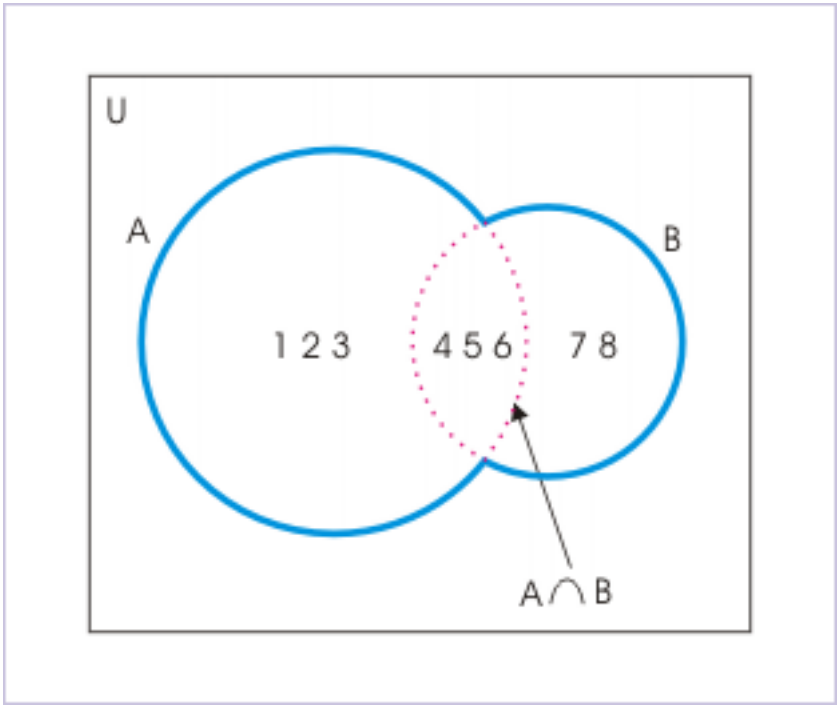

Figure 2: The intersection set consists of elements common to two sets.

1 Interpretation of Intersection set

Let us examine the defining set of intersection :

$$A \cap B = \{x : \quad x \in A \quad and \quad x \in B\}$$

We consider an arbitrary element, say "x", of the intersection set. Then, we interpret the conditional meaning as :

$$If \quad x \in A \cap B \Rightarrow x \in A \quad and \quad x \in B.$$

The conditional statement is true in opposite direction as well. Hence,

$$If x \in A \quad and \quad x \in B \Rightarrow x \in A \cap B.$$

We summarize two statements with two ways arrow as :

$$x \in A \cap B \Leftrightarrow x \in A \quad and \quad x \in B$$

In addition to two ways relation, there is an interesting aspect of intersection. Intersection is subset of either of two sets. From Venn diagram, it is clear that :

Intersection of two sets

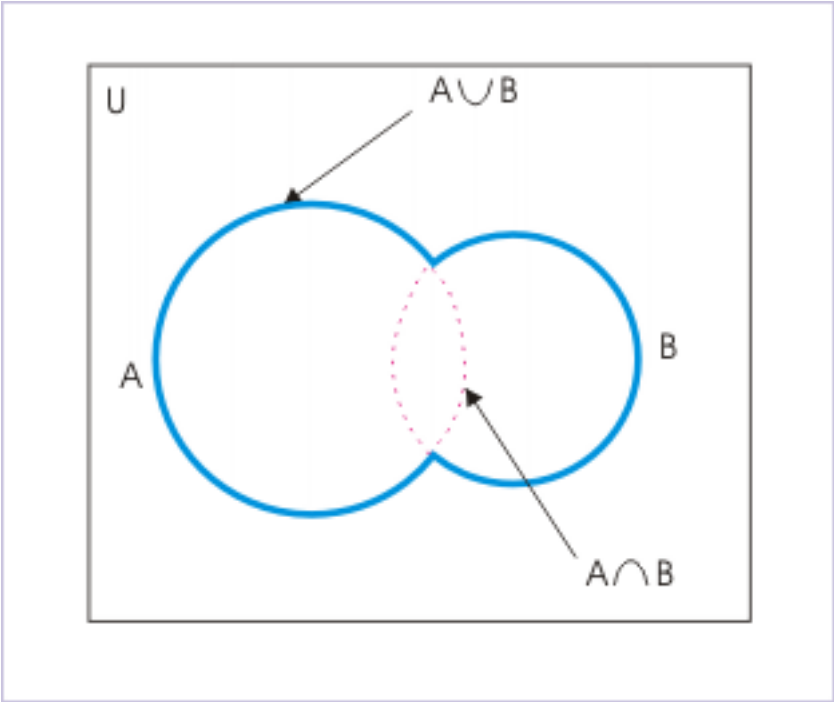

Figure 3: The intersection set consists of elements common to two sets.

$$(A \cap B) \subset A$$

and

$$(A \cap B) \subset B$$

1.1 Intersection with a subset

Since all elements of a subset is present in the set, it emerges that intersection with subset is subset. Hence, if "A" is subset of set "B", then :

$$B \cap A = A$$

1.2 Intersection of disjoint sets

If no element is common to two sets "A" and "B" , then the resulting intersection is an empty set :

$$A \cap B = \phi$$

In that case, two sets "A" and "B" are "disjoint" sets.

1.3 Multiple intersections

If $A_1, A_2, A_3, \ldots\ldots\ldots, A_n$ is a finite family of sets, then their intersections one after another is denoted as :

$$A_1 \cap A_2 \cap A_3 \cap \ldots\ldots\ldots \cap An$$

2 Important results

In this section we shall discuss some of the important characteristics/ deductions for the intersection operation.

2.1 Idempotent law

The intersection of a set with itself is the set itself.

$$A \cap A = A$$

This is because intersection is a set of common elements. Here, all elements of a set is common with itself. The resulting intersection, therefore, is set itself.

2.2 Identity law

The intersection with universal set yields the set itself. Hence, universal set functions as the identity of the intersection operator.

$$A \cap U = A$$

It is easy to interpret this law. Only the elements in "A" are common to universal set. Hence, intersection, being the set of common elements, is set "A".

2.3 Law of empty set

Since empty set is element of all other sets, it emerges that intersection of an empty set with any set is an empty set (empty set is only common element between two sets).

$$\phi \cap A = \phi$$

2.4 Commutative law

The order of sets around intersection operator does not change the intersection. Hence, commutative property holds in the case of intersection operation.

$$A \cap B = B \cap A$$

2.5 Associative law

The associative property holds with respect to intersection operator.

$$(A \cap B) \cap C = A \cap (B \cap C)$$

The intersection of sets “A” and “B” on Venn’s diagram is :

Intersection of two sets

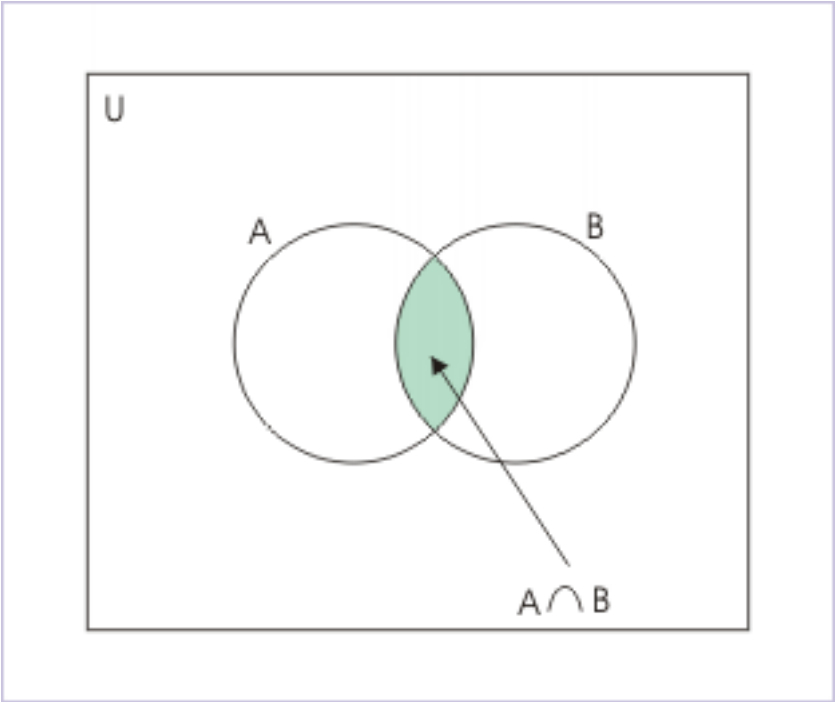

Figure 4: The intersection is a set of common elements and shown as colored region.

In turn, the intersection of set "A ∩ B" and set "C" is the small region in the center :

Intersection inloving three sets

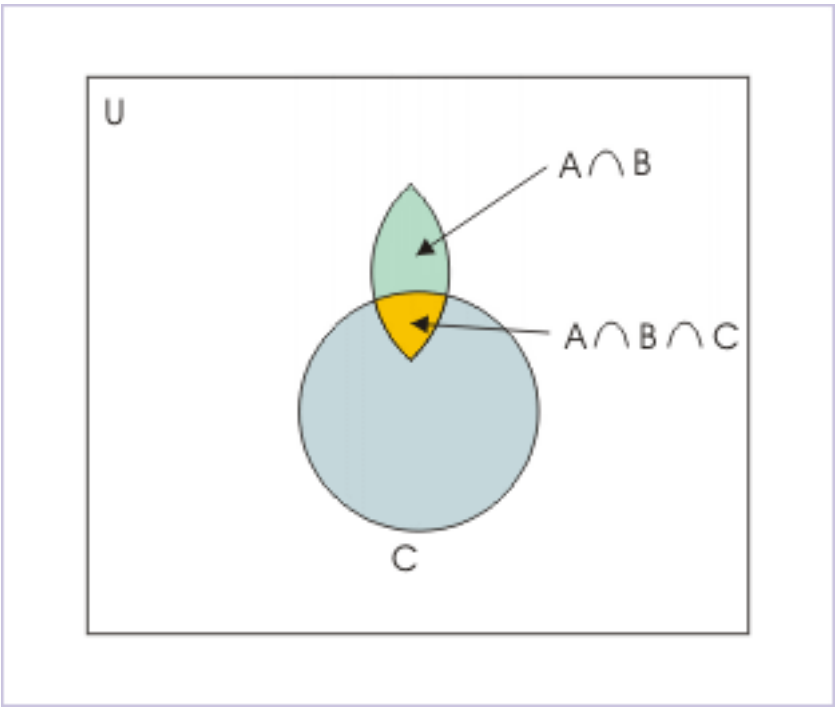

Figure 5: Intersection of a set with "the intersection set of two sets"

It is easy to visualize that the ultimate intersection is independent of the sequence of operation.

2.6 Distributive law

The intersection operator($\cap$) is distributed over union operator ($\cup$) :

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

We can check out this relation with the help of Venn diagram. For convenience, we have not shown the universal set. In the first diagram on the left, the colored region shows the union of sets “B” and “C” ie. $B \cup C$. The colored region in the second diagram on the right shows the intersection of set “A” with the union obtained in the first diagram i.e. $B \cup C$.

Distributive law

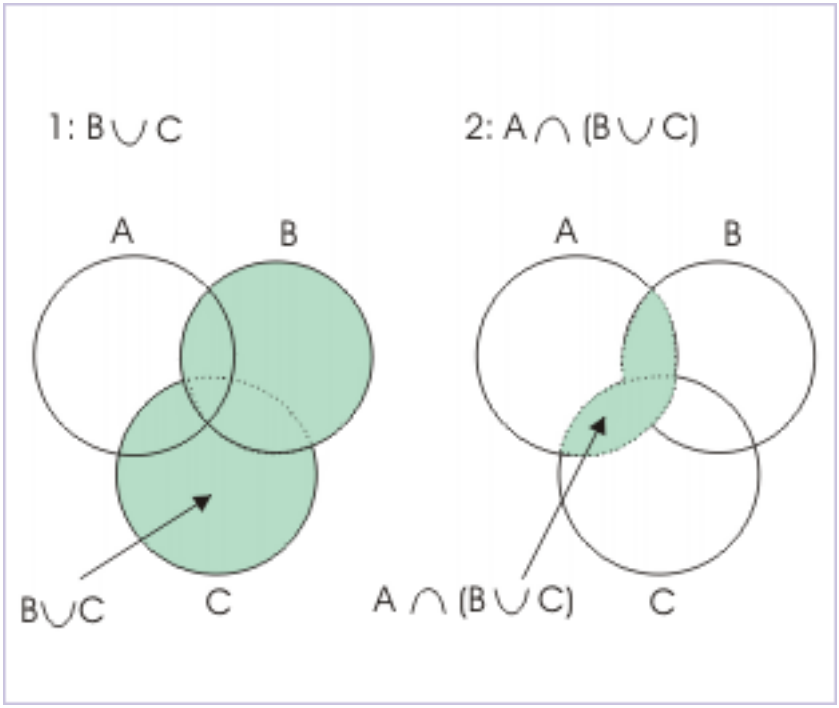

Figure 6: Distribution of intersection operator over union operator

We can now interpret the colored region in the second diagram from the point of view of expression on the right hand side of the equation :

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

The colored region is indeed the union of two intersections : " $A \cup B$" and " $A \cup C$" . Thus, we conclude that distributive property holds for "intersection operator over union operator".

In the same manner, we can prove distribution of “union operator over intersection operator” :

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

2.6.1 Analytical proof

Distributive properties are important and used for practical application. In this section, we shall prove the same in analytical manner. For this, let us consider an arbitrary element “x”, which belongs to set " $A \cap (B \cup C)$" :

$$x \in A \cap (B \cup C)$$

Then, by definition of intersection :

$$\Rightarrow x \in A \quad and \quad x \in (B \cup C)$$

$$\Rightarrow x \in A \quad and \quad (x \in B \quad or \quad x \in C)$$

$$\Rightarrow (x \in A \quad and \quad x \in B) \quad or \quad (x \in A \quad and \quad x \in C)$$

$$\Rightarrow (x \in A \cap B) \quad or \quad (x \in A \cap C)$$

$$\Rightarrow x \in (A \cap B) \quad or \quad (A \cap C)$$

$$\Rightarrow x \in (A \cap B) \cup (A \cap C)$$

But, we had started with " $A \cap (B \cup C)$ " and used its definition to show that "x" belongs to another set. It means that the other set consists of the elements of the first set – at the least. Thus,

$$\Rightarrow A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$$

Similarly, we can start with " $(A \cap B) \cup (A \cap C)$ " and reach the conclusion that :

$$\Rightarrow (A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$$

If sets are subsets of each other, then they are equal. Hence,

$$\Rightarrow A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Proceeding in the same manner, we can also prove other distributive property of "union operator over intersection operator" :

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$