Function operations*

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The form of "function as a rule" suggests that we may think of carrying out arithmetic operations like addition, multiplication etc with two functions. If we limit ourselves to real function, then we can attach meaning to equivalent of arithmetic operations with predictable domain intervals. We should, however, clearly understand that function operations with real functions involve new domain for the resulting function. In general, function operation results in contraction of intervals in which new rule formed from algebraic operation is valid.

As pointed out, function operations are defined for a new domain, depending on the type of operations - we carry out. In the nutshell, we may keep following aspects in mind, while describing function operations :

- Function operations are defined for real functions.
- The result of function operation is itself a real function.
- New function resulting from function operation is defined in new interval(s) of real numbers as determined by the nature of operation involved.

1 Domain of resulting function

The function operations, like addition, involve more than one function. Each function has its domain in which it yields real values. The resulting domain will depend on the way the domain intervals of two or more functions interact. In order to understand the process, let us consider two functions " y_1 " and " y_2 " as given below.

$$y_1 = \sqrt{(x^2 - 3x + 2)}$$

$$y_2 = \frac{1}{\sqrt{(x^2 - 3x - 4)}}$$

Let " D_1 " and " D_2 " be their respective domains. Now, the expressions in the square roots need to be non-negative. For the first function :

$$x^2 - 3x + 2 \geq 0$$

$$\Rightarrow (x - 1)(x - 2) \geq 0$$

The sign diagram is shown here. The domain for the function is the intervals in which function value is non-negative.

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Domain interval

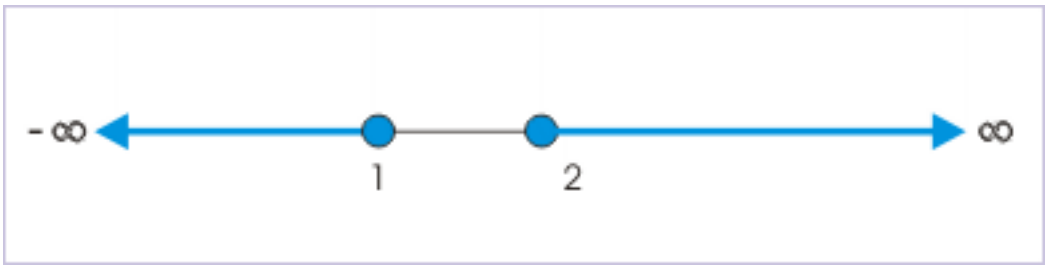

Figure 1: Domain interval of the first function

$$\Rightarrow D_1 = -\infty < x \leq 1 \quad \text{or} \quad 2 \leq x < \infty$$

Note that domain, here, includes end points as equality is permitted by the inequality "greater than or equal to" inequality. In the case of second function, square root expression is in the denominator. Thus, we exclude end points corresponding to roots of the equation.

$$x^2 - 3x - 4 > 0$$

$$\Rightarrow (x+1)(x-4) > 0$$

Domain interval

Figure 2: Domain interval of the second function

$$\Rightarrow D_2 = -\infty < x < -1 \quad \text{or} \quad 4 < x < \infty$$

Now, let us define addition operation for the two functions as,

$$y = y_1 + y_2$$

The domain, in which this new function is defined, is given by the common interval between two domains obtained for the individual functions. Here, domain for each function is shown together one over other for easy comparison.

Domain intervals

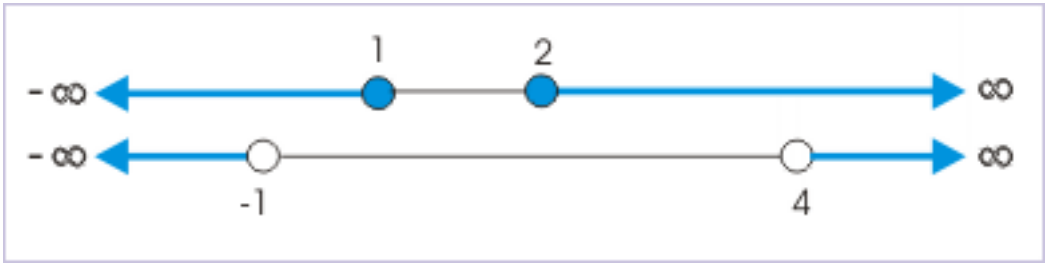

Figure 3: Domain intervals of two functions

For new function defined by addition operation, values of x should be such that they simultaneously be in the domains of two functions. Consider for example, x = 0.75. This falls in the domain of first function but not in the domain of second function. It is, therefore, clear that domain of new function is intersection of the domains of individual functions. The resulting domain of the function resulting from addition is shown in the figure.

Domain interval

Figure 4: Domain interval of the sum of two functions

$$\Rightarrow D = D_1 \cap D_2$$

This illustration shows how domains interact to form a new domain for the new function when two functions are added together.

2 Function operations

We, now, define valid operations for real functions in the light of discussion about the domain in the previous section. For different function operations, let us consider two real functions "f" and "g" with domains " D_1 " and " D_2 " respectively. Clearly, these domains are real number set R or subsets of R :

$$D_{1,}D_2 \in R$$

2.1 Addition

The addition of two real functions is denoted as "f+g". It is defined as :

$$f+g : D_1 \cap D_2 \rightarrow R \quad \text{such that :}$$

$$(f+g)(x) = f(x) + g(x) \quad \text{for all} \quad x \in D_1 \cap D_2$$

2.2 Subtraction

The subtraction of two real functions is denoted as "f-g". It is defined as :

$$f-g : D_1 \cap D_2 \rightarrow R \quad \text{such that :}$$

$$(f-g)(x) = f(x) - g(x) \quad \text{for all} \quad x \in D_1 \cap D_2$$

2.3 Scalar Multiplication

Scalar, here, means a real number constant, say "a". The scalar multiplication of a real function with a constant is denoted as "af". It is defined as :

$$af : D_1 \rightarrow R \quad \text{such that :}$$

$$(af)(x) = af(x) \quad \text{for all} \quad x \in D_1$$

2.4 Multiplication

The product of two real functions is denoted as "fg". It is defined as :

$$fg : D_1 \cap D_2 \rightarrow R \quad \text{such that :}$$

$$(fg)(x) = f(x)\, g(x) \quad \text{for all} \quad x \in D_1 \cap D_2$$

2.5 Quotient

The quotient of two real functions is denoted as "f/g". It involves rational form as "f(x)/g(x)", which is defined for g(x) $\neq$ 0. We need to exclude value of "x" for which g(x) is zero. Hence, it is defined as :

$$\frac{f}{g} : D_1 \cap D_2 - \{x | g(x) \neq 0\} \rightarrow R \quad \text{suchthat :}$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \text{for all} \quad x \in D_1 \cap D_2 - \{x | g(x) \neq 0\}$$

3 Example

Example 1

Problem : Let two functions be defined as :

$$f(x) = \sqrt{x}$$

$$g(x) = x^2 - 5x + 6$$

Find domains of "fg" and "f/g". Also define functions fg(x) and (f/g)(x).

Solution : The function f(x) is defined for all non negative real number. Hence, its domain is :

$$\Rightarrow x \geq 0$$

The function, g(x), - being a real quadratic polynomial - is real for all real values of "x". Hence, its domain is "R". Domains of two functions are shown in the figure.

Domain intervals

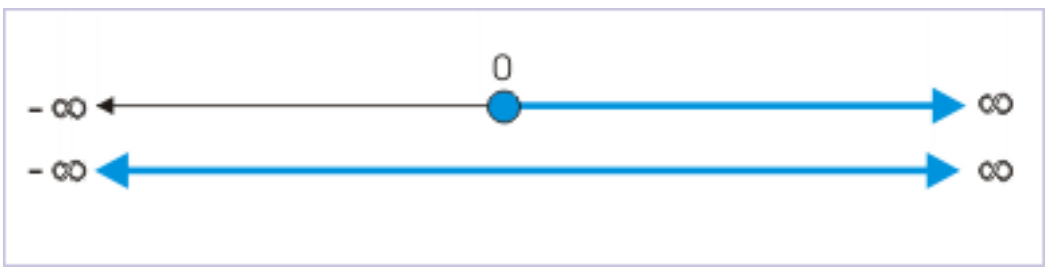

Figure 5: Domain interval of two functions

The domain of fg(x) is intersection of two intervals, which is non-negative interval as shown in the figure :

Domain interval

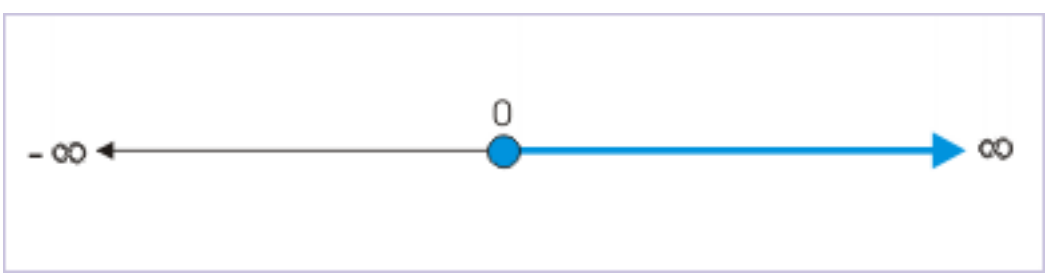

Figure 6: Domain interval of product

$$D = D_1 \cap D_2 = [0, \infty] \cap R$$

Here, we recall that intersection of a set with subset is equal to subset :

$$D = [0, \infty]$$

This is the domain of product function "fg(x)". The domain of quotient function "f/g(x)" excludes values of "x" for which "g(x)" is zero. In other words, we exclude roots of "g(x)" from domain. Now,

$$g(x) = x^2 - 5x + 6 = 0$$

$$\Rightarrow g(x) = (x-2)(x-3) = 0$$

$$\Rightarrow x = 2, 3$$

Hence, domain of "f/g(x)" is :

Domain interval

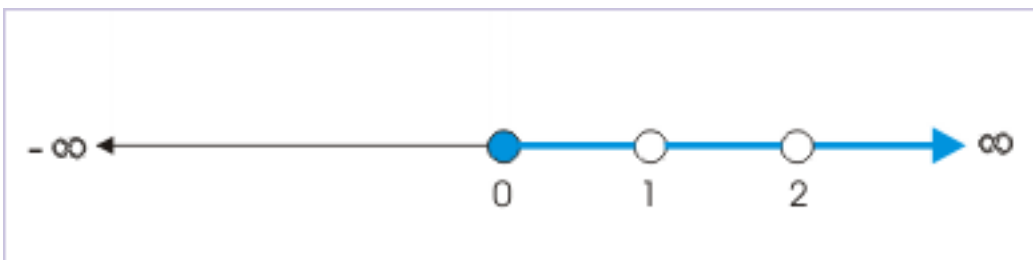

Figure 7: Domain interval of product

$$\Rightarrow D_1 \cap D_2 - \{x | g(x) \neq 0\} = [0, \infty] - \{2, 3\}$$

Now the product function, in rule form, is given as :

$$fg(x) = \sqrt{x}\left(x^2 - 5x + 6\right); \quad x \geq 0$$

Similarly, quotient function, in rule form, is given as :

$$fg(x) = \frac{\sqrt{x}}{x^2 - 5x + 6}; \quad x \geq 0, x \neq 2, x \neq 3$$

Example 2
Problem : Find domain of the function :

$$f(x) = 2\sqrt{(x-1)} + \sqrt{(1-x)} + \sqrt{(x_2 + x + 1)}$$

Solution : Given function can be considered to be addition of three separate function. We know that scalar multiplication of a function does not change domain. As such, domain of $2\sqrt{(x-1)}$ is same as that of $\sqrt{(x-1)}$. For $\sqrt{(x-1)}$ and $\sqrt{(1-x)}$,

$$x - 1 \geq 0 \quad \Rightarrow x \geq 1$$

$$1 - x \geq 0 \quad \Rightarrow x \leq 1$$

Now, we use sign rule for third function :

$$x^2 + x + 1 \geq 0$$

Here, coefficient of "x^2 " is positive and D is negative. Hence, function is positive for all real x. This means f(x)>0. This, in turn, means f(x)≥0. The domain of third function is R. Domain of given function is intersection of three domains. From figure, it is clear that only x=1 is common to three domains. Therefore,

Domain of function

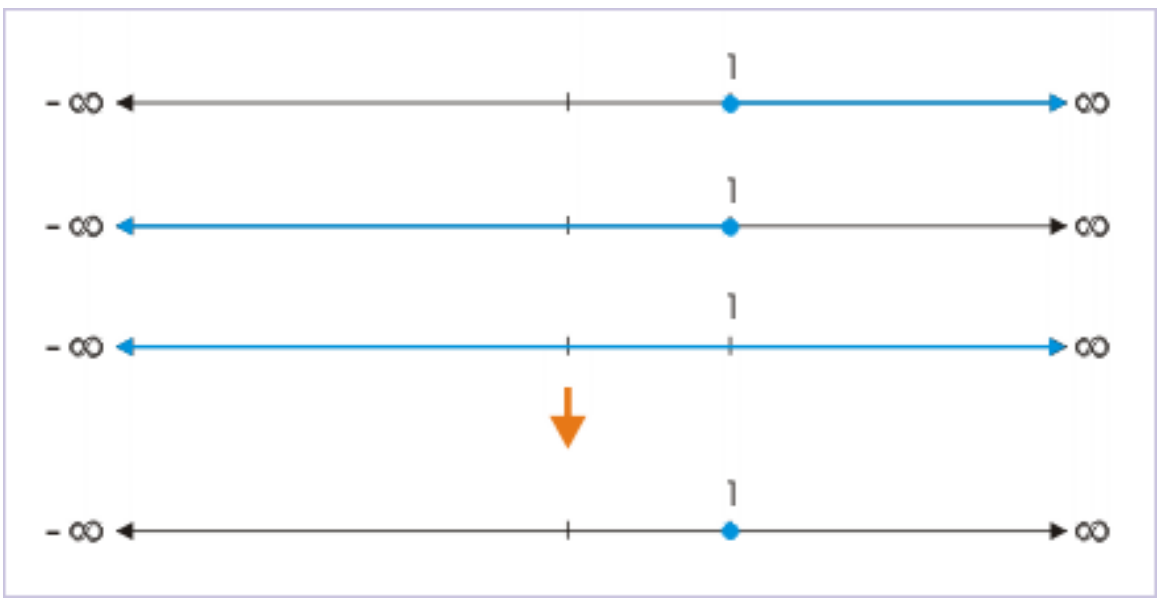

Figure 8: Intersection of three domains.

$$\text{Domain} = \{1\}$$

4 Exercise

Exercise 1 ***(Solution on p. 8.)***

Problem : Find the domain of the function given by :

$$f(x) = \frac{x}{\sqrt{(x^2 - 5x + 6)}}$$

Solutions to Exercises in this Module

Solution to Exercise (p. 7)

Solution :

The function is in rational form. We can treat numerator and denominator functions separately as f(x) and g(x). The numerator is valid for all real values of "x". Hence, its domain is "R".

$$D_1 = R$$

For determining domain of g(x), we are required to find the value of "x" for which square root in the denominator is real and not equal to zero. Thus, we need to evaluate square root expression for positive number. It means that :

$$x^2 - 5x + 6 > 0 \quad \Rightarrow (x-2)(x-3) > 0$$

The roots of the corresponding quadratic equation is 2,3. Further, coefficient of "x^2 " term is a positive number (1>0) . Therefore, intervals on the sides are positive for the quadratic expression. The valid interval satisfying the inequality is :

$$x < 2 \quad \text{or} \quad x > 3$$

$$D_2 = (-\infty, 2) \quad \cup \quad (3, \infty)$$

Now, given function is quotient of two functions. Hence, domain of the given function is intersection of two domain excluding interval that renders denominator zero. However, we have already taken into account of this condition, while determining domain of the function in the denominator. Hence,

Domain

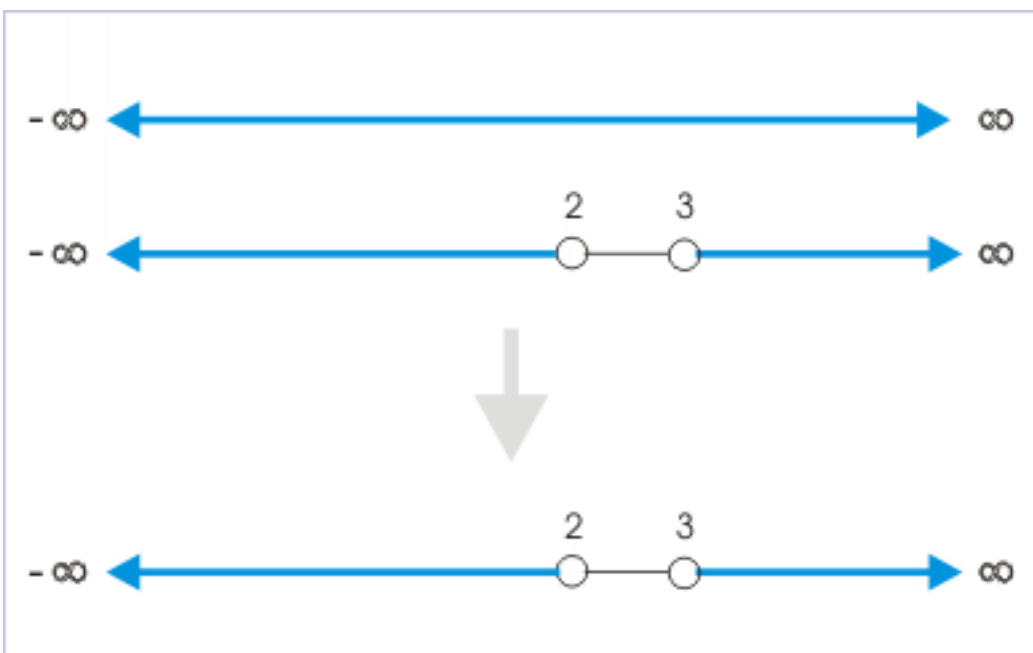

Figure 9: Domain of the function is equal to intersection of two domains.

$$\text{Domain} = D_1 \quad \cup \quad D_2 = R \quad \cap \quad \{(-\infty, 2) \quad \cup \quad (3, \infty)\}$$

$$\Rightarrow \text{Domain} = \{(-\infty, 2) \quad \cup \quad (3, \infty)\}$$