Exponential and logarithmic functions*

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Logarithmic and exponential functions are closely related functions. Logarithmic functions are useful in interpreting expressions/ equations, in which exponents are unknown. On the other hand, exponential functions are representation of natural process or mathematical relations, having exponential growth or decay. We shall encounter many expressions, involving these two functions in mathematics, while analyzing or describing processes. Further, two function types have simple derivatives and related integration results (we shall learn about derivatives and integration in calculus).

Symbolically, exponential and logarithmic functions, respectively, are :

$$f(x) = a^x$$

and

$$f(x) = \log_a x$$

In this module, we shall find out that exponential and logarithmic functions are inverse of each other. The roles of function "f(x)" and independent variable "x" are exchanged in two function definition. We shall clarify this point subsequently.

1 Exponential function

An exponential function relates every real number "x" to the exponentiation, " a^x ". In other words, we can say that an exponential function associates every real number (x) to a function given by :

$$f(x) = y = a^x$$

where :

1. The base "a" is positive real number, but excluding "1". Symbolically, $a > 0, a \neq 1$.
2. The exponent "x" is a real number.
3. The number "y" represents the result of exponentiation, " a^x " and is a positive real number. Symbolically, y >0.

We need not memorize nature of "x","y" and "a". Rather we can investigate the same by reasoning. The base "a" can not be zero, because 0^x is not uniquely defined as required for invertible function. The exponential function is designed to be invertible. This means that "f(x)" and "x" be uniquely related. Also,

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if "a" is a negative number like "-2", then $(-2)^x$ may evaluate to positive or negative value depending on whether "x" is even or odd integer. Clearly sign of function depends on the nature of integer values of "x". In case, "x" is not an integer, then sign of $(-2)^x$ can not be interpreted for all values of "x". Further if a=1, then 1^x evaluates to "1" for all values of "x". Again function is not uniquely defined. In the nutshell, we conclude that base "a" is a positive number, but not equal to "1". In particular, the exponential function corresponding to base "e", which is equal to 2.718281828, is called "natural" exponential function.

Once nature of "a" is decided, it is easy to find the nature of "y". Consider simplified exponents like 10^2 , 10^1 , 10^{-2}, 10^{-100}. All these numbers are greater than zero. This is true even if exponent is not integer. We conclude that "y" is positive number. Note that neither "a" nor "y" are zero. On the other hand, "x" by definition is a real number.

Clearly, the domain and range of exponential function are :

$$\text{Value of "x"} = \text{Domain} = R$$

$$\text{Value of "y"} = \text{Range} = (0, \infty)$$

We shall see subsequently that roles of "x" and "f(x)" are exchanged for logarithmic function. Here, "x" is the exponent i.e. the logarithmic value of a positive number and "f(x)" is the result of exponentiation, which is argument of logarithmic function. For this reason, we say that exponential and logarithmic functions are inverse to each other. As expected for inverse functions, we shall also see that domain and range of exponential and logarithmic functions are exchanged.

The nature of exponential function are different around a=1. The plots of exponential functions for two cases (i)$0 < a < 1$ and (ii) $a > 1$ are discussed here. If the base is greater than zero, but less than "1", then the exponential function asymptotes to positive x-axis. It is easy to visualize the nature of plot. It is placed in the positive upper part as "f(x)" is positive. Also, note that $(0.25)^2$ is greater than $((0.25)^4$. Hence, plot begins from a higher value to lower value as "x" increases, but never becomes equal to zero.

Exponential function

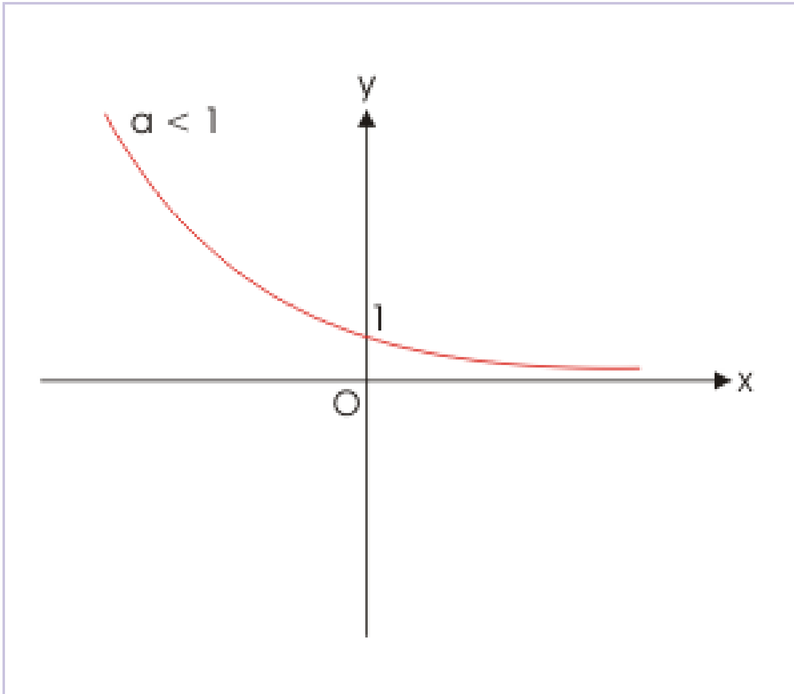

Figure 1: The plot of exponential function, when base is less than "1".

If the base is greater than “1”, then the exponential function asymptotes to negative x-axis. Again, it is easy to visualize the nature of plot. It is placed in the positive upper part as "f(x)" is positive. Also, note that $(1.25)^2$ is less than $((1.25)^4$. Hence, plot begins from a lower value to higher value as "x" increases.

Exponential function

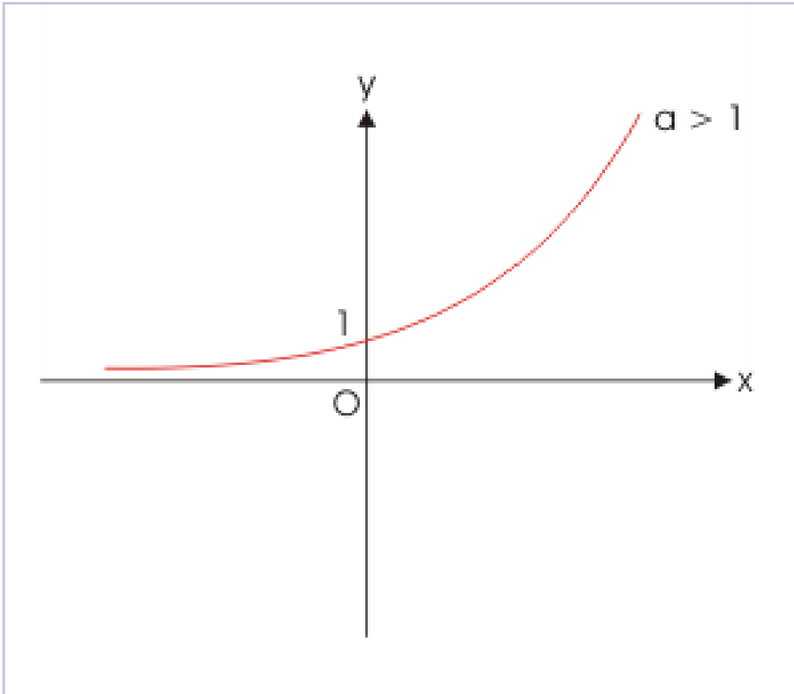

Figure 2: The plot of exponential function, when base is greater than "1".

Note that expanse of exponential function is along x – axis on either side of the y-axis, showing that its domain is R. On the other hand, the expanse of "y" is limited to positive side of y-axis, showing that its range is positive real number. Further, irrespective of base values, all plots intersect y-axis at the same point i.e. y = 1 as :

$$y = a^x = a^0 = 1$$

2 Logarithmic functions

A logarithmic function gives "exponent" of an expression in terms of a base, "a", and a number, "x". The following two representations, in this context, are equivalent :

$$a^y = x$$

and

$$f\left(x\right) = y = \log_a x$$

where :

1. The base "a" is positive real number, but excluding "1". Symbolically, $a > 0, a \neq 1$.
2. The number "x" represents result of exponentiation, " a^y " and is also a positive real number. Symbolically, x >0.
3. The exponent "y" i.e. logarithm of "x" is a real number.

Note that neither “a” nor “x” equals to zero.

The expression of a logarithm for “x” on a certain base represents logarithmic function. In words, we can say that a logarithmic function associates every positive real number (x) to a real valued exponent (y), which is symbolically represented as :

$$f\left(x\right) = y = \log_a x; \quad a, x > 0, \quad a \neq 1$$

Following earlier discussion for the case of exponential function, we exclude "a = 1" as logarithmic function is not relevant to this base.

$$1^y = 1$$

We can easily see here that whatever be the exponent, the value of logarithmic function is “1". Hence, base “1” is irrelevant as exponent “y” is not uniquely associated with “x”.

From the defining values of "x" and "f(x)", we conclude that domain and range of logarithmic function is :

$$\text{Value of “x”} = \text{Domain} = (0, \infty)$$

$$\text{Value of “y”} = \text{Range} = R$$

Note that domain and range of logarithmic function is exchanged with respect to domain and range of exponential function.

2.1 Base

The base of the logarithmic function can be any positive number. However, “10” and “e” are two common bases that we often use. Here, “e” is a mathematical constant given by :

$$e = 2.718281828$$

If we use “e” as the base, then the corresponding logarithmic function is called “natural” logarithmic function. The plots, here, show logarithmic functions for two bases (i) 10 and (ii) e.

Logarithmic function

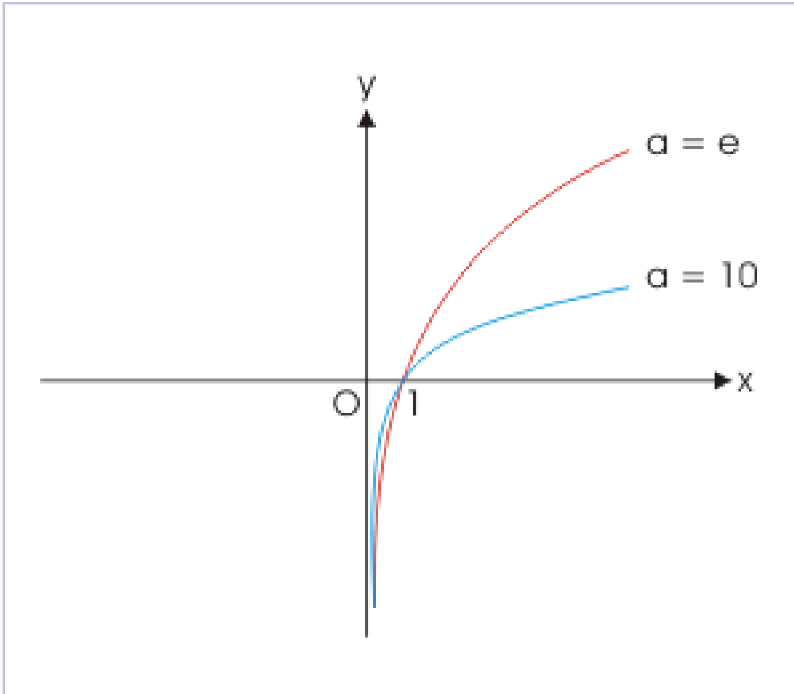

Figure 3: The plots of logarithmic function on different bases.

Note that expanse of logarithmic function "f(x)" is along y – axis on either side of axis, showing that its range is R. On the other hand, the expanse of “x” is limited to positive side of x-axis, showing that domain is positive real number. Further, irrespective of bases, logarithmic plots intersect x-axis at the same point i.e. x = 1 as :

$$x = a^y = a^0 = 1$$

2.2 Graphs

We have noted the importance of base “1” for logarithmic function. Base “1” plays an important role for determining nature of logarithmic function. Here, we have drawn plots for two cases : (i) $0<a<1$ and (ii) $a>1$. If the base is greater than zero, but less than “1”, then the logarithm function asymptotes to positive y-axis. Since, "x" is positive number, plot falls only in the right side quadrants. Also, note that $\log_{0.1}0.001 = 3$ is less than $\log_{0.1}0.01 = 2$. Hence, plot begins from a higher value to lower value as "x" increases. Further, $\log_{0.1}10 = -1$. We deduce that plot moves below x-axis for "x" greater than "1".

Logarithmic function

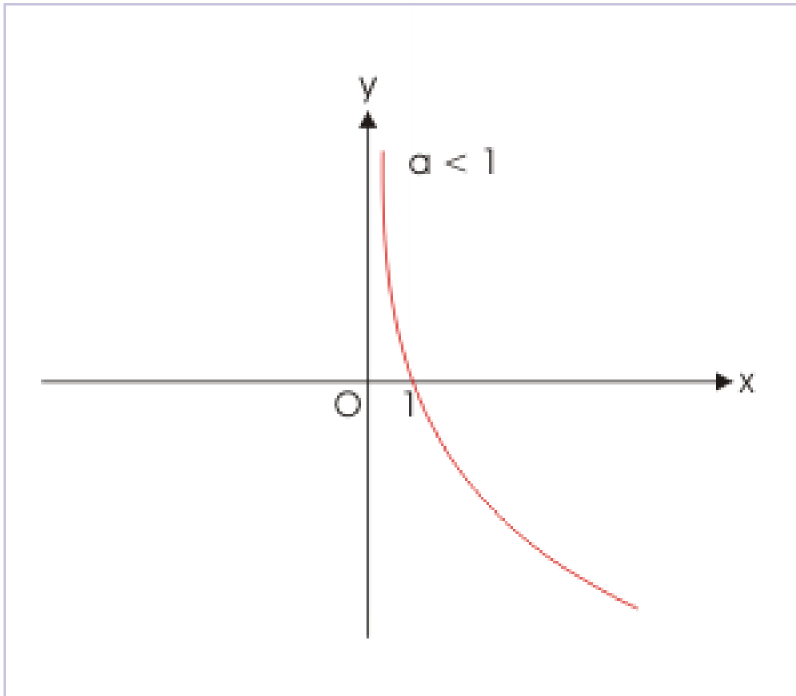

Figure 4: The plot of logarithmic function, when base is less than "1".

If the base is greater than “1”, then the logarithm function asymptotes to negative y-axis. Since, "x" is positive number, plot again falls only in the right side quadrants. Also, note that $\log_{10} 10 = 1$ is greater than $\log_{10} 100 = 2$. Hence, plot begins from a lower value to greater value as "x" increases. Further, $\log_{10} 10 = 1$. We deduce that plot moves above x-axis for "x" greater than "1".

Logarithmic function

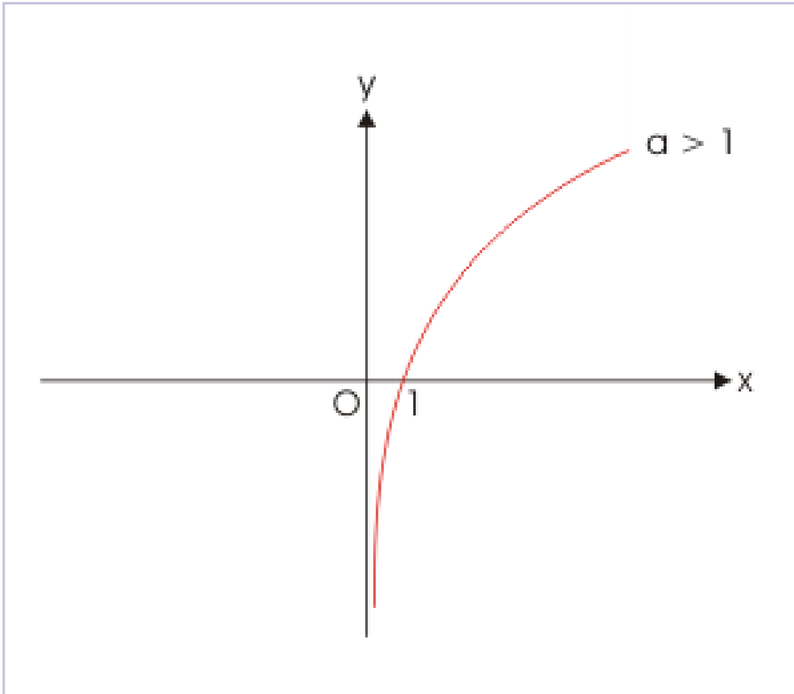

Figure 5: The plot of logarithmic function, when base is greater than "1".

2.3 Logarithmic identities

Some of the important logarithmic identities are given here without proof. Idea, here, is to simply equip ourselves so that we can work with logarithmic functions in conjunction with itself or with other functions.

$$\log_a a = 1$$

$$\log_a 1 = 0$$

$$x = a^y = a^{\log_a x}$$

$$a^{\log_b c} = c^{\log_b a}$$

$$\log_a (xy) = \log_a (x) + \log_a (y)$$

$$\log_a \left(\frac{x}{y}\right) = \log_a (x) - \log_a (y)$$

$$\log_a (x^y) = y\log_a (x)$$

$$\log_a\left(x^{\frac{1}{y}}\right) = \frac{\log_a(x)}{y}$$

2.3.1 Change of base

Sometimes, we are required to work with logarithmic expressions of different bases. In such cases, we convert them to same base, using following relation :

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

Example 1

Problem : Find the domain of the function given by :

$$f(x) = \frac{1}{\log_{10}(1-x)}$$

Solution : The given function is reciprocal of a logarithmic function. Therefore, we first need to ensure that logarithmic function does not evaluate to zero for a value of "x". A logarithmic function evaluates to zero if its argument is equal to "1". Hence,

$$1 - x \neq 1$$

$$\Rightarrow x \neq 0$$

Further domain of logarithmic function is a positive real number. For that :

$$1 - x > 0$$

$$\Rightarrow 1 > x$$

$$\Rightarrow x < 1$$

Combining two results, the domain of the given function is :

$$\text{Domain of “f”} = (-\infty, 1) - \{0\}$$

Example 2

Problem : Find all real values of "x" such that :

$$1 - e^{\frac{1}{x}-1} > 0$$

Solution : Solving for the exponential function, we get following inequality :

$$\Rightarrow e^{\frac{1}{x}-1} < 1$$

Now, we know that domain of an exponential function is "R". However, this information is not helpful here to find values of "x" that satisfies the inequality. Taking natural logarithm on either side of the equation,

$$\Rightarrow \frac{1}{x} - 1 < \log_e 1$$

But, logarithm of "1" i.e. $\log_e 1$ is zero. Hence,

$$\Rightarrow \frac{1}{x} - 1 < 0$$

$$\Rightarrow \frac{1-x}{x} < 0$$

This is rational inequality. In order to solve this inequality, we need to multiply each side of the inequality by -1 so that "x" in "1-x" becomes positive. This multiplication changes the inequality to "greater than" inequality.

$$\Rightarrow \frac{x-1}{x} > 0$$

Here, critical points are 0 and 1. The rational expression is positive in the intervals on either side of middle interval.

Sign convention

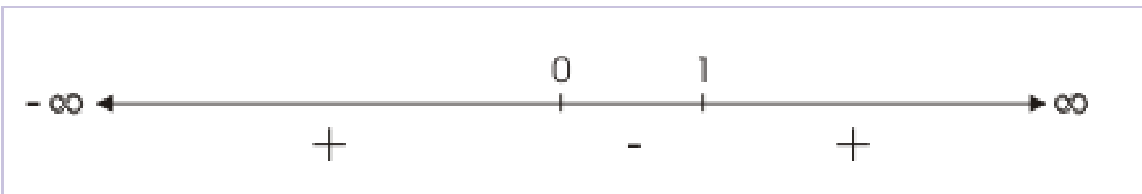

Figure 6: Positive and negative intervals.

Picking the intervals for which expression is positive :

Intervals

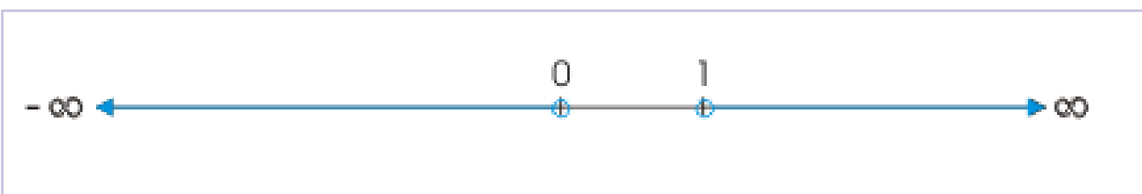

Figure 7: Intervals for which rational function is positive.

$$\text{Domain} = (-\infty, 0) \cup (1, \infty)$$

3 Logarithmic inequality

The nature of logarithmic function is dependent on the base value. We know that base of a logarithmic function is a positive number excluding "1". The value of "1" plays important role in deciding nature of logarithmic function and hence that of inequality associated to it. Let us consider an equality :

$$\log_a x > y$$

What should we conclude : $x > a^y$ or $x < a^y$? It depends on the value of "a". We can understand the same by considering LHS of the inequality equal to an exponent z :

$$\log_a x = z$$

If a > 1, then "a^z" will yield a greater "x" than "a^y", because z>y (it is given by the inequality). On the other hand, if 0<a<1, then "a^z" will yield a smaller "x" than "a^y", because z>y. We can understand this conclusion with the help of an example. Let

$$\log_2 x > 3$$

Let

$$\log_2 x = 4$$

Clearly, $2^4 > 2^3$ as $16 > 8$
Let us now consider a <1,

$$\log_{0.5} x > 3$$

Let

$$\log_{0.5} x = 4$$

Clearly, $0.5^4 < 0.5^3$ as $0.0625 < 0.125$. Thus, we finally conclude :

$$\log_a x > y \Leftrightarrow x > a^y; \quad a > 1$$

$$\log_a x > y \Leftrightarrow x < a^y; \quad 0 < a < 1$$

We have used two ways notation to indicate that interpretation of logarithmic inequality is true in either direction. Similarly, we can conclude that :

$$\log_a x < y \Leftrightarrow x < a^y; \quad a > 1$$

$$\log_a x < y \Leftrightarrow x > a^y; \quad 0 < a < 1$$

Logarithmic inequality involving logarithmic functions on same base on either side of the inequality can be interpreted as :

$$\log_a x > \log_a y \Leftrightarrow x > y; \quad a > 1$$

$$\log_a x > \log_a y \Leftrightarrow x < y; \quad 0 < a < 1$$

Alternate method

We know that if x>y, then :

$$a^x > a^y; \quad a > 1 \quad \text{and} \quad a^x < a^y; \quad 0 < a < 1$$

Let us now consider the inequality $\log_a x > y$. Following above deduction, we use the term of inequality as powers on a common base :

$$a^{\log_a x} > a^y; \quad a > 1 \quad \text{and} \quad a^{\log_a x} < a^y; \quad 0 < a < 1$$

Using identity, $x = a^{\log_a x}$,

$$x > a^y; \quad a > 1 \quad \text{and} \quad x < a^y; \quad 0 < a < 1$$

Similarly, we can deduce other results of logarithmic inequalities.