Inverse trigonometric functions*

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Inverse trigonometric function returns an angle corresponding to a real number, following certain rule. They are inverse functions corresponding to trigonometric functions. The inverse function of sine , for example, is defined as :

$$f\left(x\right)=\sin^{-1}x;\quad x\in\left[-1,1\right]$$

where "x" is a real number, "f(x)" is the angle. Clearly, "f(x)" is the angle, whose sine is "x". Symbolically,

$$\Rightarrow \sin\{f\left(x\right)\}=\sin\{\sin^{-1}\left(x\right)\}=x$$

In the representation of inverse function, we should treat "-1" as symbol – not as power. In particular,

$$\sin^{-1}\left(x\right)\neq\frac{1}{\sin x}$$

Inverse trigonometric functions are also called arc functions. This is an alternative notation. The corresponding functions are arcsine, arccosine, arctangent etc. For example,

$$\Rightarrow f\left(x\right)=\sin^{-1}\left(x\right)=\arcsin\left(x\right)$$

1 Nature of trigonometric functions

Trigonometric functions are many-one relations. The trigonometric ratio of different angles evaluate to same value. If we draw a line parallel to x-axis such that 0 < y < 1, then it intersects sine plot for multiple times – ,in fact, infinite times. It follows, then, that we can associate many angles to the same sine value. The trigonometric functions are, therefore, not an injection and hence not a bijection. As such, we can not define an inverse of trigonometric function in the first place! We shall see that we need to redefine trigonometric functions in order to make them invertible.

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sine function

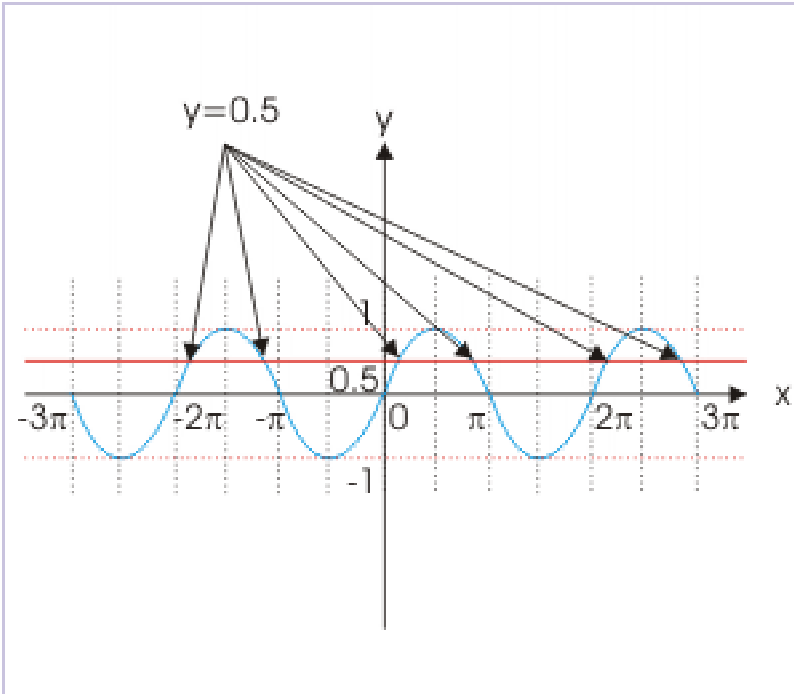

Figure 1: many-one relation

In order to define, an inverse function, we require to have one-one relation in both directions between domain and range. The function needs to be a bijection. It emerges that we need to shorten the domain of trigonometric functions such that a distinct angle corresponds to a distinct real number. Similarly, a distinct real number corresponds to a distinct angle.

We can identify many such shortened intervals for a particular trigonometric function. For example, the shortened domain of sine function can be any one of the intervals defined by :

$$\left[(2n-1)\frac{\pi}{2},(2n+1)\frac{\pi}{2}\right], \quad n \in Z$$

The domain corresponding to n = 0 yields principal domain given by :

$$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

The nature of trigonometric functions is periodic. Same values repeat after certain interval. Here, our main task is to identify an interval of "x" such that all possible values of a trigonometric function are included once. This will ensure one-one relation in both directions between domain and range of the function. This interval is easily visible on graphs of the corresponding trigonometric function.

2 Inverse trigonometric functions

Every angle in the new domain (shortened) is related to a distinct real number in the range. Inversely, every real number in the range is related to a distinct angle in the domain of the trigonometric function. We are aware that the elements of the "ordered pair" in inverse relation exchanges their places. Therefore, it follows that domain and range of trigonometric function are exchanged for corresponding inverse function i.e. domain becomes range and range becomes domain.

2.1 arcsine function

The arcsine function is inverse function of trigonometric sine function. From the plot of sine function, it is clear that an interval between $-\pi/2$ and $\pi/2$ includes all possible values of sine function only once. Note that end points are included. The redefinition of domain of trigonometric function, however, does not change the range.

sine function

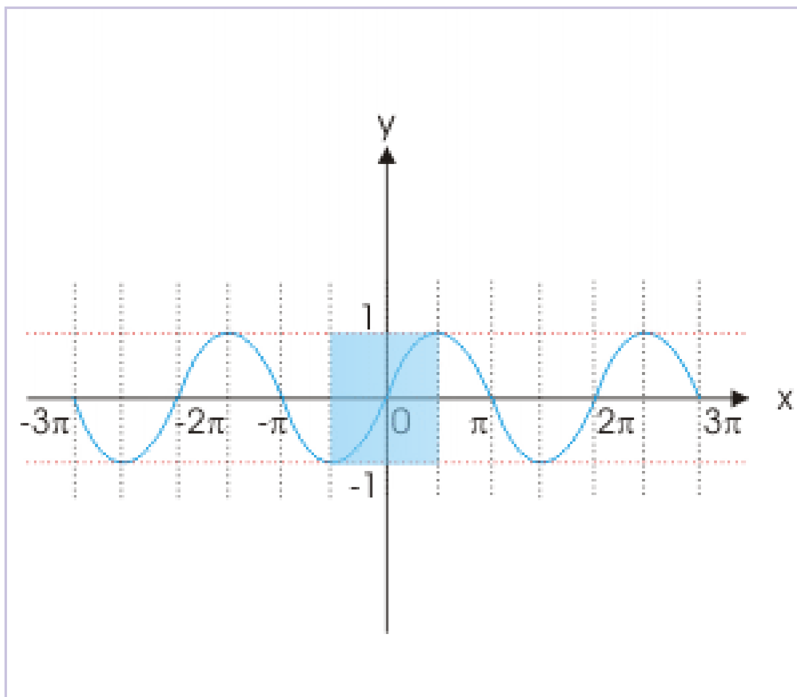

Figure 2: Redefined domain of function

$$\text{Domain of sine} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\text{Range of sine} = [-1, 1]$$

This redefinition renders sine function invertible. Clearly, the domain and range are exchanged for the inverse function. Hence, domain and range of the inverse function are :

$$\text{Domain of arcsine} = [-1, 1]$$

$$\text{Range of arcsine} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Therefore, we define arcsine function as :

$$f : [-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad \text{by f(x) = arcsin(x)}$$

The arcsin(x) .vs. x graph is shown here.

arcsine function

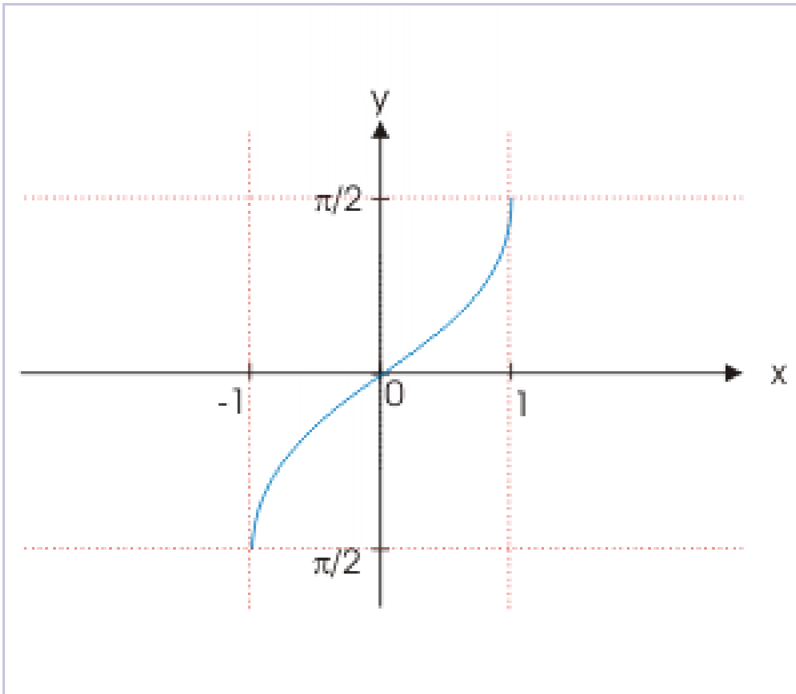

Figure 3: The arcsine function .vs. real value

2.2 arccosine function

The arccosine function is inverse function of trigonometric cosine function. From the plot of cosine function, it is clear that an interval between 0 and π includes all possible values of cosine function only once. Note that end points are included. The redefinition of domain of trigonometric function, however, does not change the range.

cosine function

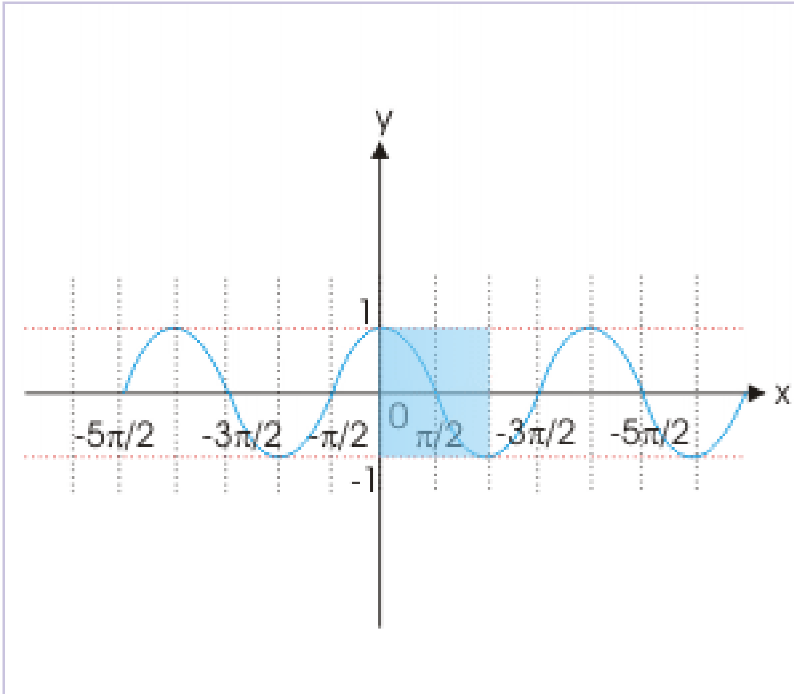

Figure 4: Redefined domain of function

$$\text{Domain of cosine} = [0, \pi]$$

$$\text{Range of cosine} = [-1, 1]$$

This redefinition renders cosine function invertible. Clearly, the domain and range are exchanged for the inverse function. Hence, domain and range of the inverse function are :

$$\text{Domain of arccosine} = [-1, 1]$$

$$\text{Range of arccosine} = [0, \pi]$$

Therefore, we define arccosine function as :

$$f : [-1, 1] \rightarrow [0, \pi] \quad \text{by f(x)} = \text{arccos(x)}$$

The arccos (x) .vs. x graph is shown here.

arccosine function

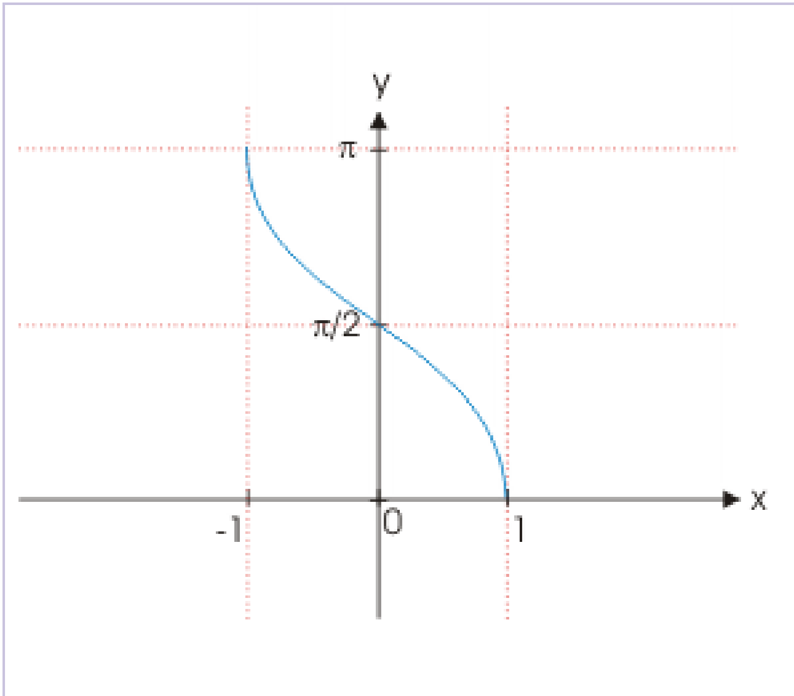

Figure 5: The arccosine function .vs. real value

2.3 arctangent function

The arctangent function is inverse function of trigonometric tangent function. From the plot of tangent function, it is clear that an interval between $-\pi/2$ and $\pi/2$ includes all possible values of tangent function only once. Note that end points are excluded. The redefinition of domain of trigonometric function, however, does not change the range.

tangent function

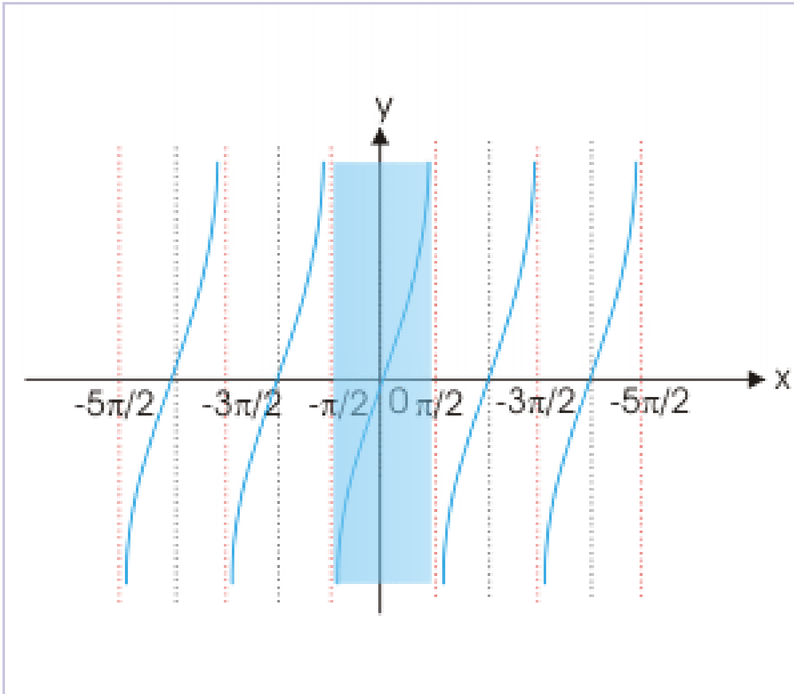

Figure 6: Redefined domain of function

$$\text{Domain of tangent} = \left(-\frac{\pi}{2, }\frac{\pi}{2}\right)$$

$$\text{Range of tangent} = R$$

This redefinition renders tangent function invertible. Clearly, the domain and range are exchanged for the inverse function. Hence, domain and range of the inverse function are :

$$\text{Domain of arctangent} = R$$

$$\text{Range of arctangent} = (-\pi/2, \pi/2)$$

Therefore, we define arctangent function as :

$$f : R \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \quad \text{by f(x) = arctan (x)}$$

The arctan(x) .vs. x graph is shown here.

arctangent function

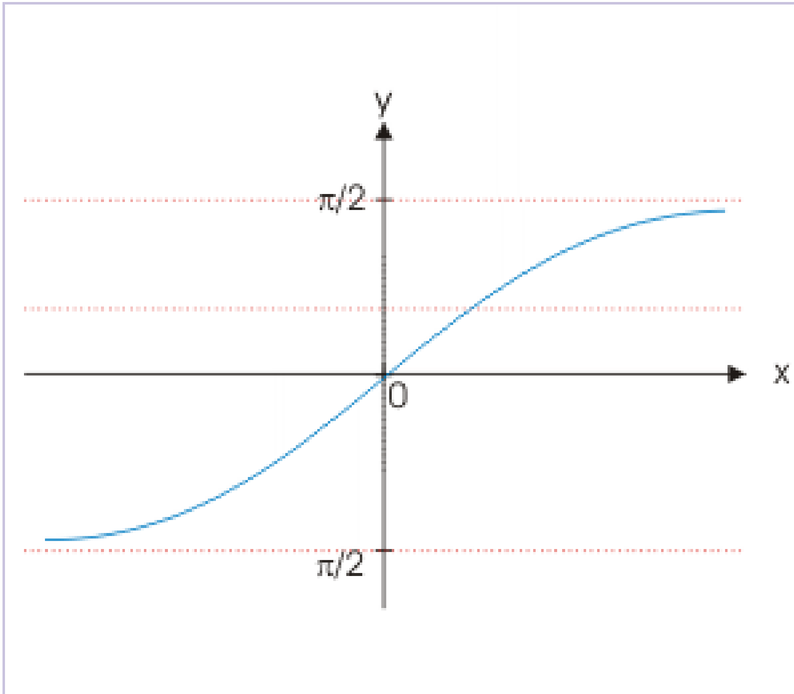

Figure 7: The arctangent function .vs. real value

2.4 arccosecant function

The arccosecant function is inverse function of trigonometric cosecant function. From the plot of cosecant function, it is clear that union of two disjointed intervals between " $-\pi/2$ and 0" and "0 and $\pi/2$ " includes all possible values of cosecant function only once. Note that zero is excluded, but " $-\pi/2$ " and " $\pi/2$ " are included . The redefinition of domain of trigonometric function, however, does not change the range.

cosecant function

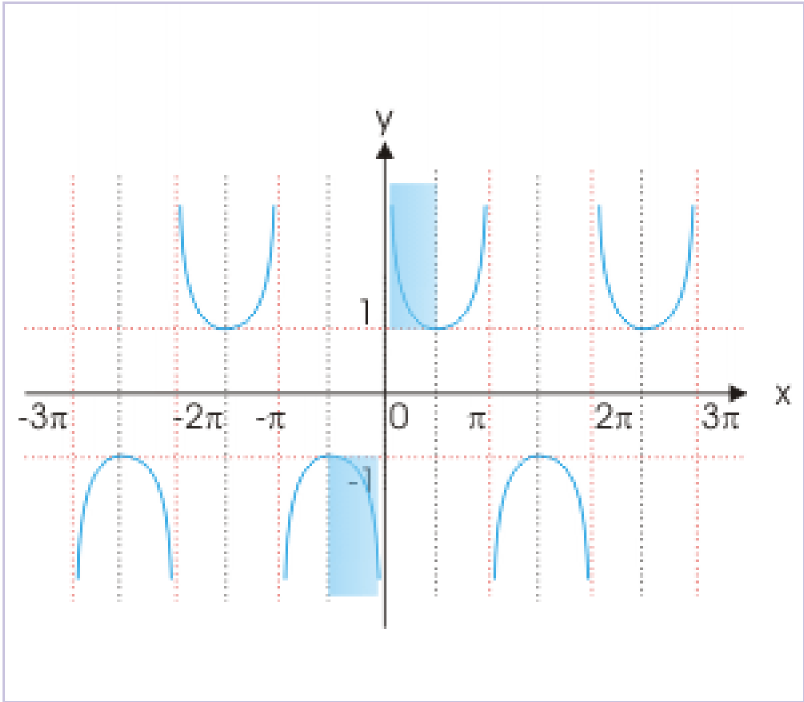

Figure 8: Redefined domain of function

$$\text{Domain of cosecant} = [-\pi/2, \pi/2] - \{0\}$$

$$\text{Range of cosecant} = (-\infty, -1] \cup [1, \infty) = R - (-1, 1)$$

This redefinition renders cosecant function invertible. Clearly, the domain and range are exchanged for the inverse function. Hence, domain and range of the inverse function are :

$$\text{Domain of arccosecant} = R - (-1, 1)$$

$$\text{Range of arccosecant} = [-\pi/2, \pi/2] - \{0\}$$

Therefore, we define arccosecant function as :

$$f : R - (-1, 1) \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\} \quad \text{by f(x)} = \text{arccosec (x)}$$

The arccosec(x) .vs. x graph is shown here.

arccosecant function

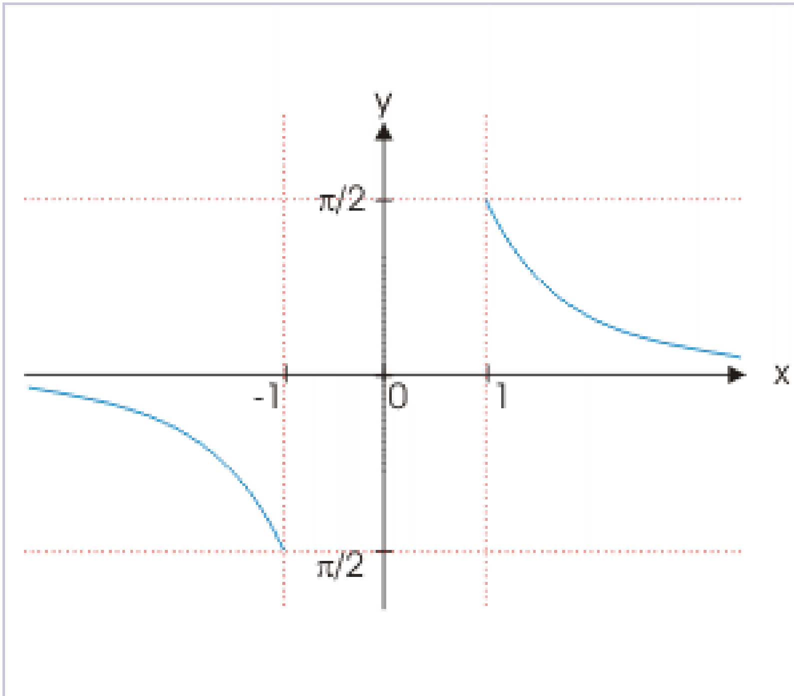

Figure 9: The arccosecant function .vs. real value

2.5 arcsecant function

The arcsecant function is inverse function of trigonometric secant function. From the plot of secant function, it is clear that union of two disjointed intervals between "0 and $\pi/2$ " and " $\pi/2$ and π " includes all possible values of secant function only once. Note that " $\pi/2$" is excluded. The redefinition of domain of trigonometric function, however, does not change the range.

secant function

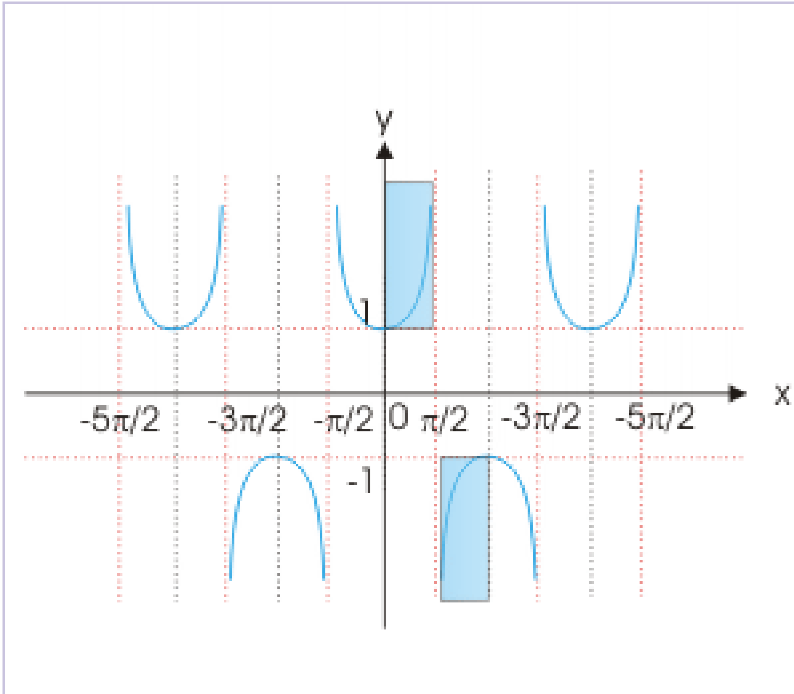

Figure 10: Redefined domain of function

$$\text{Domain of secant} = [0, \pi/2) \cup (\pi/2, \pi] = [0, \pi] - \{\pi/2\}$$

$$\text{Range of secant} = (-\infty, -1] \cup [1, \infty) = R - (-1, 1)$$

This redefinition renders secant function invertible. Clearly, the domain and range are exchanged for the inverse function. Hence, domain and range of the inverse function are :

$$\text{Domain of arcsecant} = R - (-1, 1)$$

$$\text{Range of arcsecant} = [0, \pi] - \{\pi/2\}$$

Therefore, we define arcsecant function as :

$$f : R - (-1, 1) \rightarrow [0, \pi] - \{\pi/2\} \quad \text{by f(x) = arcsec (x)}$$

The arcsec(x) .vs. x graph is shown here.

arcsecant function

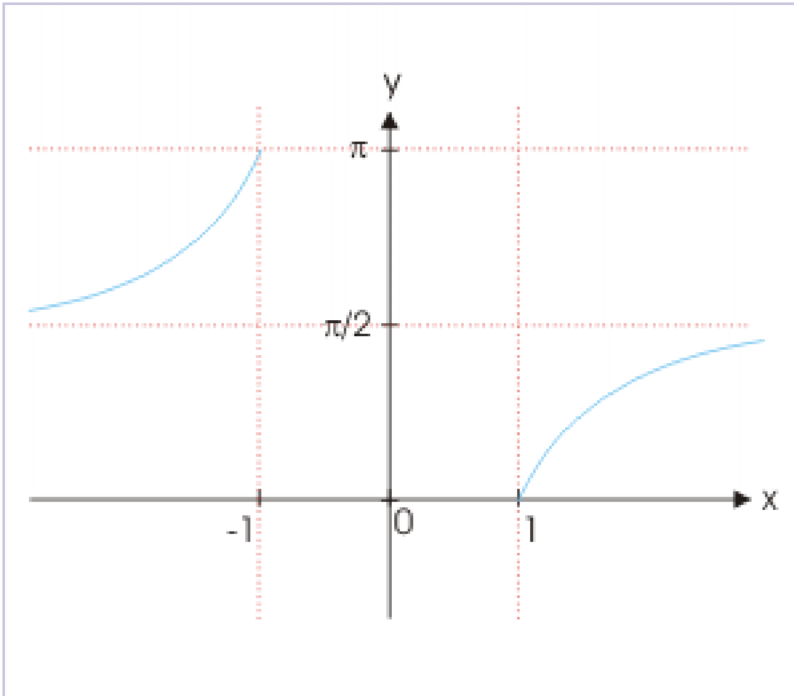

Figure 11: The arcsecant function .vs. real value

2.6 arccotangent function

The arccotangent function is inverse function of trigonometric cotangent function. From the plot of cotangent function it is clear that an interval between 0 and π includes all possible values of cotangent function only once. Note that end points are excluded. The redefinition of domain of trigonometric function, however, does not change the range.

cotangent function

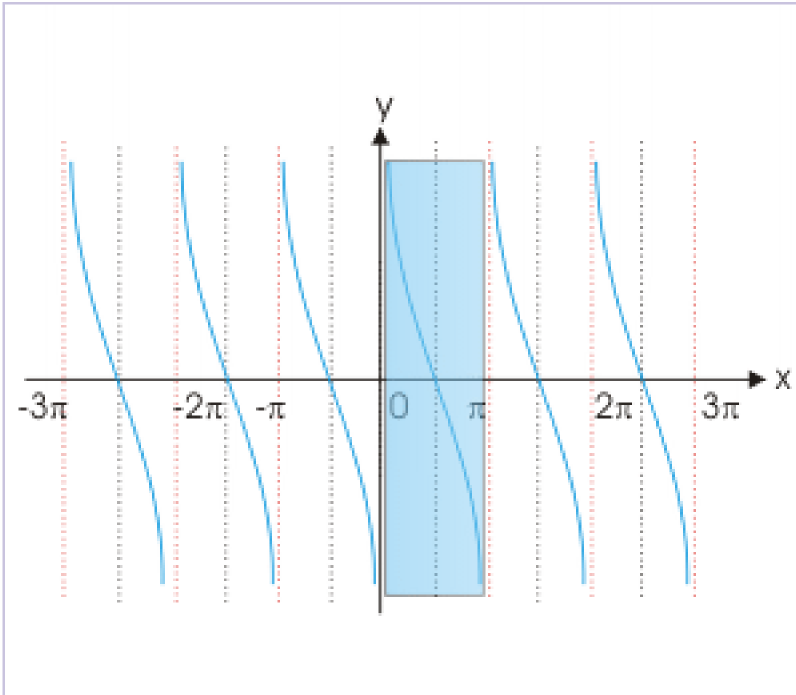

Figure 12: Redefined domain of function

$$\text{Domain of cotangent} = (0, \pi)$$

$$\text{Range of cotangent} = R$$

This redefinition renders cotangent function invertible. Clearly, the domain and range are exchanged for the inverse function. Hence, domain and range of the inverse function are :

$$\text{Domain of arccotangent} = R$$

$$\text{Range of arccotangent} = (0, \pi)$$

Therefore, we define arccotangent function as :

$$f : R \rightarrow (0, \pi) \quad \text{by f(x) = arccot (x)}$$

The arccot(x) .vs. x graph is shown here.

arccotangent function

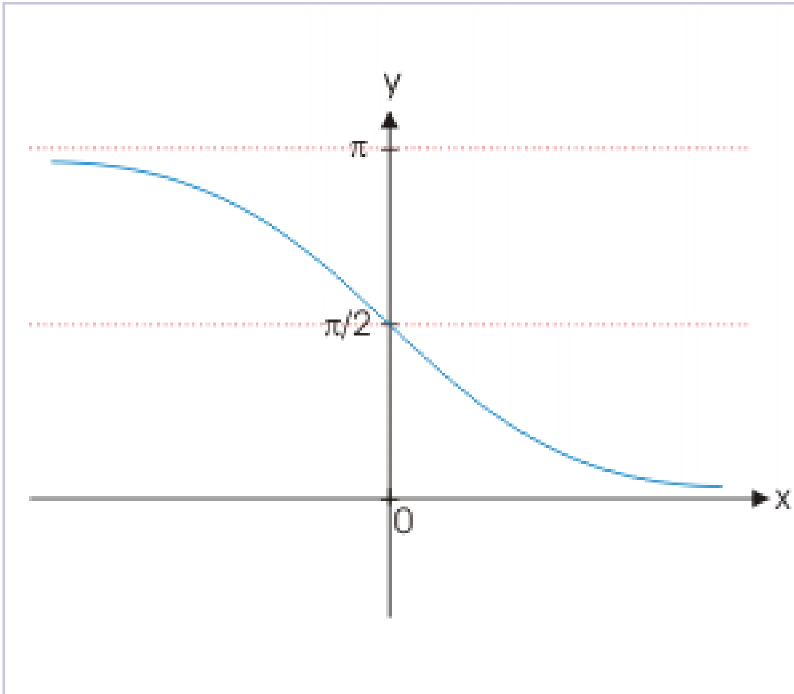

Figure 13: The arccotangent function .vs. real value

3 Example

Example 1

Problem : Find y when :

$$y = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$

Solution : There are multiple angles for which :

$$\Rightarrow \tan y = x = -\frac{1}{\sqrt{3}}$$

However, range of sine function is [-$\pi/2$, $\pi/2$]. We need to find angle, which falls in this range. Now, acute angle corresponding to the value of $1/\sqrt{3}$ is $\pi/6$. In accordance with sign diagram, tangent is negative in second and fourth quarters. But range is [-$\pi/2$, $\pi/2$]. Hence, we need to find angle in fourth quadrant. The angle in the fourth quadrant whose tangent has magnitude of $1/\sqrt{3}$ is given by :

$$\Rightarrow y = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

Corresponding negative angle is :

$$\Rightarrow y = \frac{11\pi}{6} - 2\pi = -\frac{\pi}{6}$$

Example 2

Problem : Find domain of the function given by :

$$f(x) = \frac{\cos^{-1}(x)}{[x]}$$

Solution : The given function is quotient of two functions having rational form :

$$f(x) = \frac{g(x)}{h(x)}$$

The domain of quotient is given by :

$$D = D_1 \cap D_2 - \{x : x \quad \text{when h(x)} = 0\}$$

Here, $g(x) = \cos^{-1}(x)$. The domain of arccosine is [-1,1]. Hence,

$$D_1 = \text{Domain of “g”} = [-1, 1]$$

The denominator function h(x) is greatest integer function. Its domain is "R".

$$D_2 = \text{Domain of “h”} = R$$

The intersection of two domains is :

$$\Rightarrow D_1 \cap D_2 = [-1, 1] \cap R = [-1, 1]$$

Intersection of domains

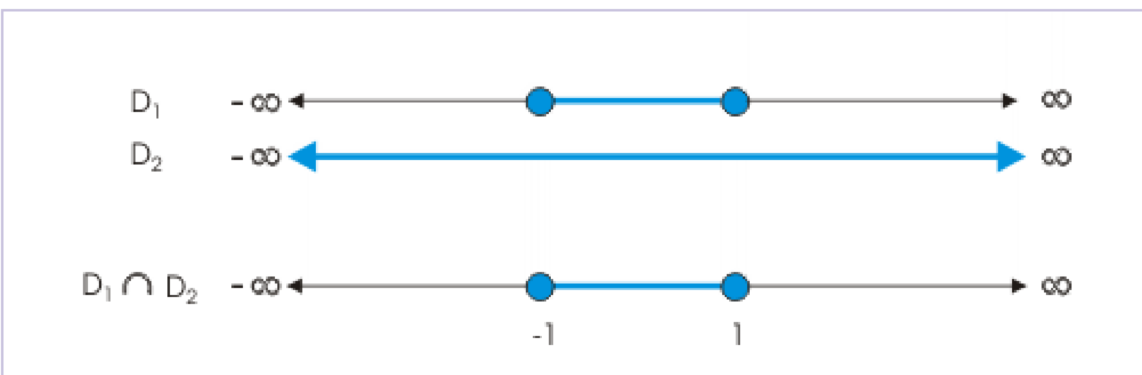

Figure 14: The intersection of domains result in common interval.

Now, greatest integer function becomes zero for values of "x" in the interval [0,1). Hence, domain of given function is :

Domain of function

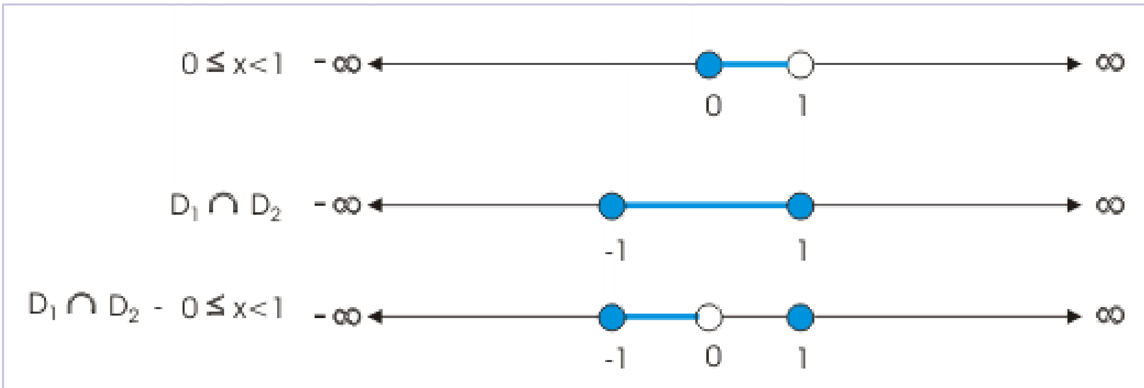

Figure 15: The domain of function is obtained by subtracting interval, which is not permitted.

$$D = D_1 \cap D_2 - [0, 1)$$

$$D = [-1, 1] - [0, 1) = -1 \le x < 0 \cup \{1\}$$

4 Summary

Redefined domains of trigonometric functions are tabulated here :

Trigonometric Function	Old Domain	New Domain	Old Range	New Range
sine	R	[-π/2, π/2]	[-1,1]	[-1,1]
cosine	R	[0, π]	[-1,1]	[-1,1]
tan	R -- odd multiples of π/2	(-π/2, π/2)	R	R
cosecant	R -- integer multiple of π	[-π/2, π/2] -- {0}	R -- (-1,1)	R -- (-1,1)
secant	R - odd multiples of π/2	[0, π] -- {π/2}	R -- (-1,1)	R -- (-1,1)
cotangent	R -- integer multiple of π	(0, π)	R	R

We observe that there is no change in the range – even though domains of the trigonometric functions have changed.

The corresponding domain and range of six inverse trigonometric functions are tabulated here.

Inverse Trigonometric Function	Domain	Range
arcsine	[-1,1]	[-π/2, π/2]
arccosine	[-1,1]	[0, π]

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arctangent         R       (-π/2, π/2)
arccosecant         R -- (-1,1)      [-π/2, π/2] -- {0}
arcsecant         R -- (-1,1)      [0, π] -- {π/2}
arccotangent         R       (0, π)
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5 Exercise

Exercise 1 *(Solution on p. 18.)*

Find the domain of the function given by :

$$f\left(x\right) = 2^{\sin^{-1}\left(x\right)}$$

Exercise 2 *(Solution on p. 18.)*

Problem : Find the domain of the function given by :

$$f\left(x\right) = \cos^{-1}\frac{3}{3 + \sin x}$$

Exercise 3 *(Solution on p. 18.)*

Find range of the function :

$$f\left(x\right) = \frac{1}{2 - \sin 2x}$$

Exercise 4 *(Solution on p. 19.)*

Find domain of function

$$f\left(x\right) = \sin^{-1}\{\log_2\left(x^2 + 3x + 4\right)\}$$

Exercise 5 *(Solution on p. 19.)*

Find the range of the function

$$f\left(x\right) = \cos^{-1}\left(\frac{x^2}{1 + x^2}\right)$$

Exercise 6 *(Solution on p. 19.)*

Find the range of the function

$$f\left(x\right) = \text{cosec}^{-1}\left[1 + \sin^2 x\right]$$

where [.] denotes greatest integer function.

Solutions to Exercises in this Module

Solution to Exercise (p. 17)

The exponent of the exponential function is inverse trigonometric function. Exponential function is real for all real values of exponent. We see here that given function is real for the values of “x” corresponding to which arcsine function is real. Now, domain of arcsine function is [-1,1]. This is the interval of "x" for which arcsine is real. Hence, domain of the given function, “f(x)” is :

$$\text{Domain} = [-1, 1]$$

Solution to Exercise (p. 17)

Solution : The given function is an inverse cosine function whose argument is a rational function involving trigonometric function. The domain interval of inverse cosine function is [-1, 1]. Hence, value of argument to inverse cosine function should lie within this interval. It means that :

$$-1 \le \frac{3}{3 + \sin x} \le 1$$

Comparing with the form of modulus, $|x| \le 1 \quad \Rightarrow -1 \le x \le 1$, we conclude :

$$\Rightarrow \frac{|3|}{|3 + \sin x|} \le 1$$

Since, modulus is a non-negative number, the inequality sign remains same after simplification :

$$\Rightarrow |3| \le |3 + \sin x|$$

Again $3 > 0$ and 3+sin x > 0, we can open up the expression within the modulus operator without any change in inequality sign :

$$\Rightarrow 3 \le 3 + \sin x \quad \Rightarrow 0 \le \sin x \quad \Rightarrow \sin x \ge 0$$

The solution of sine function is the domain of the given function :

$$\text{Domain} = 2n\pi \le x \le (2n + 1)\,\pi, \quad x \in Z$$

Solution to Exercise (p. 17)

We have already solved this problem by building up interval in earlier module. Here, we shall find domain conventionally by solving for x. The denominator of given function is non-negative as value of sin2x can not exceed 1. Hence, domain of function is real number set R. Further, maximum value of sin2x is 1. Hence,

$$y = f\left(x\right) = \frac{1}{2 - \sin 2x} > 1$$

This means given function is positive for all real x. Now, solving for x,

$$\Rightarrow 2y - y\sin 2x = 1$$

$$\Rightarrow \sin 2x = \frac{2y - 1}{y}$$

$$\Rightarrow x = \frac{1}{2}\sin^{-1}\left(\frac{2y - 1}{y}\right)$$

We know that domain of sine inverse function is [-1,1]. Hence,

$$-1 \le \frac{2y-1}{y} \le 1$$

Since y>0, we can simplify this inequality as :

$$-y \le 2y-1 \le y$$

Either,

$$\Rightarrow 2y-1 \ge -y$$

$$\Rightarrow y \ge \frac{1}{3}$$

Or,

$$\Rightarrow 2y-1 \le y$$

$$\Rightarrow y \le 1$$

$$\text{Range} = \left[\frac{1}{3}, 1\right]$$

Solution to Exercise (p. 17)

This is a composite function in which quadratic function is argument of logarithmic function. The logarithmic function is, in turn, argument of inverse sine function. In such case, it is advantageous to evaluate from outer to inner part. The domain of outermost inverse trigonometric function is [-1,1].

$$-1 \le \{\log_2\left(x^2+3x+4\right)\} \le 1$$

$$\log_2 2^{-1} \le \{\log_2\left(x^2+3x+4\right)\} \le \log_2 2$$

$$\frac{1}{2} \le \left(x^2+3x+4\right) \le 2$$

For the first inequality,

$$\Rightarrow 2x^2+6x+8 \ge 1$$

$$\Rightarrow 2x^2+6x+7 \ge 0$$

This quadratic function is positive for all value of x. For the second inequality,

$$\Rightarrow x^2+3x+4 \le 2$$

$$\Rightarrow x^2+3x+2 \le 0$$

The solution of this inequality is [1,2]. The intersection of R and [1,2] is [1,2]. Hence, domain of given function is [1,2].

Solution to Exercise (p. 17)

Hint : The range of rational expression as argument of inverse trigonometric function is [0,1]. But, domain of arccosine is [-1,1] and range is [0,π]. The function is continuously decreasing. The maximum and minimum values are 0 and1 (see arccosine graph). Hence, range of given function is [0, $\pi/2$].

Solution to Exercise (p. 17)

The minimum and maximum value of $\sin^2 x$ is 0 and 1. Hence, range of

$$1 + \sin^2 x$$

is defined in the interval given by :

$$1 \le 1 + \sin^2 x \le 2$$

The corresponding values returned by GIF are 1 and 2. It means :

$$\left[1 + \sin^2 x\right] = \{1, 2\}$$

But domain of arccosecant is [-π/2, π/2] – {0}. Refer graph of arccosecant. Thus, arccosecant can take only 1 as its argument, which falls within the domain of arccosecant. Hence, range of given function is a singleton :

$$\text{Range} = \{\text{cosec}^{-1} 1\}$$