Rational function*

Sunil Kumar Singh

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Rational function is defined in similar fashion as rational number is defined in terms of numerator and denominator. Implicitly, we refer "real" rational function here. It is defined as the ratio of two real polynomials with the condition that polynomial in the denominator is not a zero polynomial.

$$f(x) = \frac{p(x)}{q(x)}; \quad q(x) \neq 0$$

Rational function is not defined for values of x for which denominator polynomial evaluates to zero as ratio "p(x)/0" is not defined. Some examples of rational function are :

$$f(x) = \frac{2x^2 - x + 1}{2x^2 - 5x - 3}; \quad x \neq -\frac{1}{2}, x \neq 3$$

$$g(x) = \frac{x + 1}{2x^2 - x + 1}$$

$$h(x) = \frac{2x^4 - x^2 + 1}{x + 1}; \quad x \neq -1$$

Note second example function, g(x) above. There is no exclusion point for this rational polynomial. The denominator polynomial is $2x^2 - x + 1$, whose determinant is negative and coefficient of x^2 term is positive. It means denominator of g(x) is positive for all values of x. We should also note that values of x being excluded are points - not a continuous interval. Further, the notation to denote exclusion is an "inequation" – not "inequality" – because notation $x \neq -1$ negates corresponding equation x = -1. Recall that inequality, on the other hand, compares relative values.

1 Domain of rational function

Domain of rational function is domain of numerator polynomial minus exclusion points as determined by zeroes of denominator polynomial. Since domain of polynomial is R, domain of rational polynomial is R minus exclusion points determined by denominator. The domains for three rational functions given above are :

$$\text{Domain of f(x)} = R - \{-\frac{1}{2}, 3\}$$

$$\text{Domain of g(x)} = R$$

$$\text{Domain of h(x)} = R - \{-1\}$$

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2 Important properties of rational function

Important properties are :

- Singularity or exception point
- Holes
- Asymptotes – vertical, horizontal and slant
- x and y intercepts

2.1 Singularities

Singularities are x-values for which denominator of rational function is zero. The function is not defined for such x-values and as such these values are excluded from the domain set of the function. These points are also called exception points. Function is not defined at these points.

Factorizing numerator and denominator of rational function helps to identify singularities of algebraic rational function. Singularities correspond to x values resulting from equating linear factors in denominator to zero. The important thing to note here is that singularity or exception occurs when denominator of rational function turns zero – no matter whether linear factor in the denominator cancels out with the linear factor in numerator or not. To understand this point, let us consider few rational functions given below :

$$f(x) = \frac{(x-1)(x+2)}{(x-1)(x+1)}$$

$$g(x) = \frac{(x-1)^2(x+2)}{(x-1)(x+1)}$$

$$h(x) = \frac{(x-1)(x+2)}{(x-1)^2(x+1)}$$

We can see that h(x) contains a linear factor (x-1) in the denominator after cancellation of like linear factors. On the other hand, functions f(x) and g(x) do not contain (x-1) in the denominator after cancellation of like linear factors. The function g(x), however, contains (x-1) in the numerator after cancellation. Notwithstanding these possibilities, denominator of the rational function turns zero at x=1. As such, the point specified by x=1 is singularity for all three function forms shown above.

We shall see that either a hole or vertical asymptote occurs at the point of exception i.e. singularity. It depends on how do the linear factors in the denominator relate to linear factors in the numerator.

Exercise 1 ***(Solution on p. 20.)***

Find singularities of function given by :

$$\frac{x^2-3x+2}{x^2-2x-3}$$

2.2 Holes

Hole exists at a singularity when corresponding linear factor of the denominator cancel out completely or when linear factor remains in the numerator after cancellation. Hole is a point on the graph where function value is not defined. It is a point having x and y coordinates. We determine x-coordinate by equating linear factor in denominator to zero. Its y-coordinate is obtained by plugging x-value in the reduced (after cancellation of common factors in numerator and denominator) function form. As pointed out, there are two situations with respect to position of a hole :

1: If linear factor cancels completely, then hole lies any where but not on x-axis.

$$f(x) = \frac{(x-1)(x+2)}{(x-1)(x+1)}$$

Here, singularities occur at x= -1 and 1. The linear factor (x-1) is present in both numerator and denominator and as such cancels out completely. Therefore, there is a hole at x=1. On the other hand, no cancellation is involved at x=-1. There exists a vertical asymptote at x=-1. Important point to realize here is that if linear factor is present in denominator only and there is no cancellation involved, then x-value corresponding to linear factor is not the x-coordinate of hole. We shall learn about vertical asymptote subsequently. Now, the y-coordinate of hole is :

$$\Rightarrow f(x) = \frac{(x+2)}{(x+1)} = \frac{3}{2} = 1.5$$

The graph of function, f(x), is shown in the figure below :

Rational function

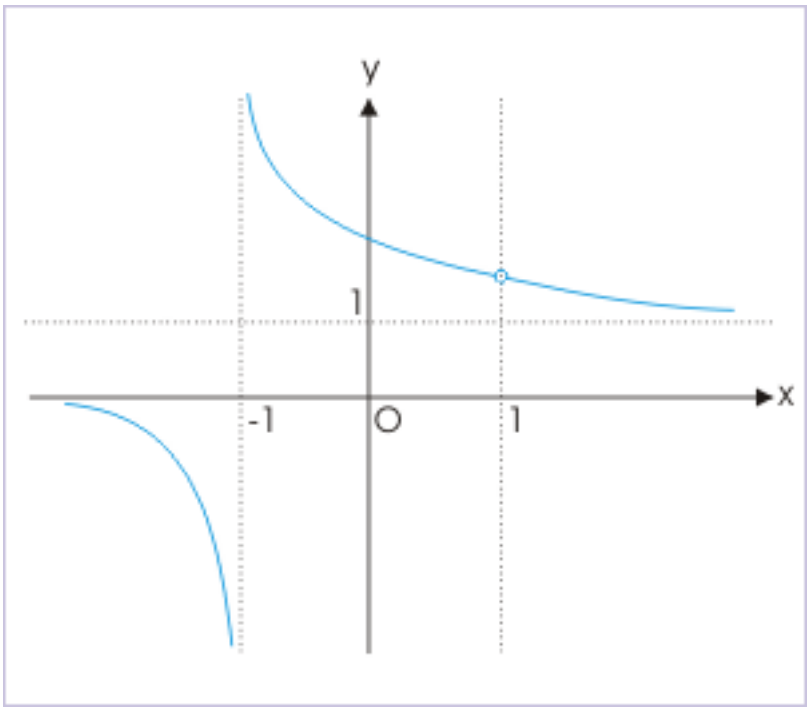

Figure 1: Singularities, holes and vertical asymptote

2: If linear factor remains in the numerator after cancellation, then hole lies on x-axis. The graph tends to intercept x-axis. As such, hole exists at x-axis.

$$g(x) = \frac{(x-1)^2(x+2)}{(x-1)(x+1)}$$

Here, singularities occur at x= -1 and 1. There is a vertical asymptote at x=-1, but a hole at x=1. The y-coordinate of hole is :

$$\Rightarrow g(x) = \frac{(x-1)(x+2)}{(x+1)} = \frac{0X3}{2} = 0$$

Thus, hole lies on x-axis.

Rational function

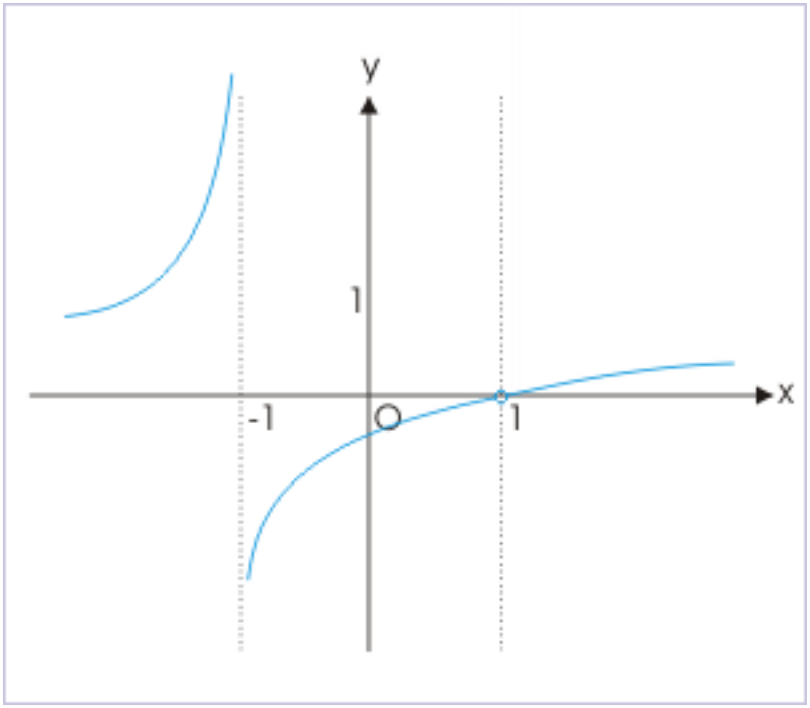

Figure 2: Singularities, holes and vertical asymptote

2.3 Asymptotes

An asymptote is a straight line at singularity which graph of function tends to approach but never touches. The difference between graph and asymptotes is infinitesimally small as the graph is extended away from x-axis. Important to note is that graph neither touches or crosses the asymptote. In the case of vertical asymptote, the function values tend to be either positive or negative large number or a combination of two on either side of the vertical asymptote. In the case of horizontal asymptote, the function values tend to be a finite value.

2.3.1 Vertical asymptote

Vertical asymptote is a vertical line including y-axis to which graph of function comes closer and closer but never touches. Vertical asymptotes correspond to very large y-values, where difference between x-value and asymptote is infinitesimally small. An equation of vertical asymptote has the form,

$$x = c$$

Vertical asymptotes occur at singularity when linear factor in the denominator remains after cancellation or otherwise. Let us investigate three functions given earlier for existence of vertical asymptote.

$$f(x) = \frac{(x-1)(x+2)}{(x-1)(x+1)}$$

$$g(x) = \frac{(x-1)^2(x+2)}{(x-1)(x+1)}$$

$$h(x) = \frac{(x-1)(x+2)}{(x-1)^2(x+1)}$$

For function, f(x), singularities exists at x=1 and -1. Here x=1 is a hole as linear factor (x-1) cancels out completely. The linear factor (x+1), however, does not cancel out. Thus, a vertical asymptote exists at x=-1. See graph shown earlier drawn for f(x).

For function, g(x), also singularities exists at x=1 and -1. Here x=1 is a hole on x-axis as linear factor (x-1) remains in the numerator after cancellation. The linear factor (x+1), however, does not cancel out. Thus, a vertical asymptote exists at x=-1. See graph shown earlier drawn for g(x).

For function, h(x), also singularities exists at x=1 and -1. Here, a vertical asymptote exists at x=1 as linear factor (x-1) remains in the denominator. The linear factor (x+1), however, does not cancel out. Thus, a vertical asymptote also exists at x=-1. See graph shown here for h(x).

Rational function

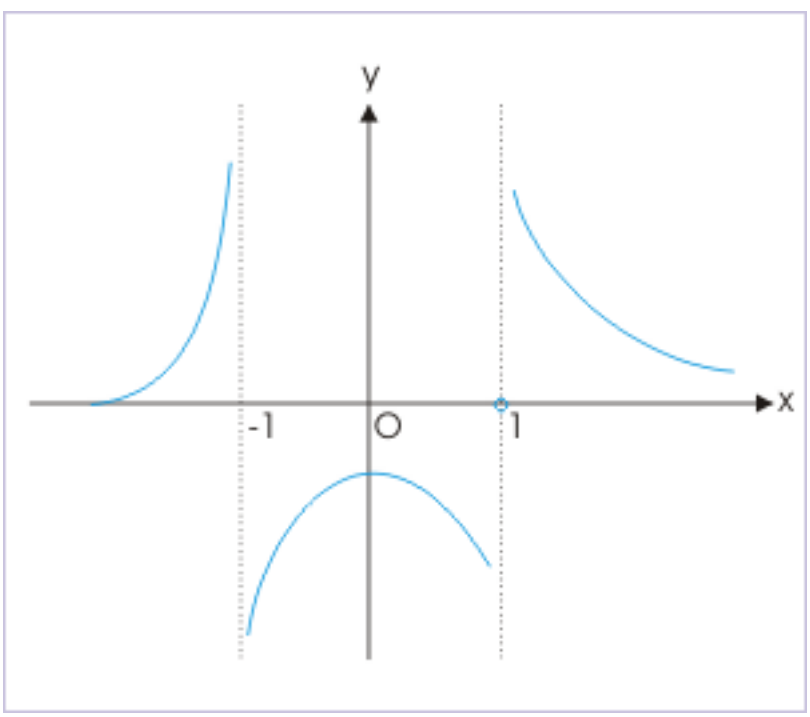

Figure 3: Singularities, holes and vertical asymptote

2.3.1.1 Nature of vertical asymptote

The function value assumes large values close to singularity where asymptote exists. The values are directed either in the same of opposite directions. It depends on the polarity of reduced function. If the reduced function has linear factor raised to even power, then values asymptotes in the same direction. On the other hand, if the reduced function has linear factor raised to odd power, then values asymptotes in opposite directions. Let us consider function as defined here,

$$f(x) = \frac{(x-1)(x+2)}{(x-1)^3(x+1)}$$

After simplification, the function reduces to :

$$\Rightarrow f(x) = \frac{(x+2)}{(x-1)^2(x+1)}$$

Clearly, (x-1) is raised to even power 2. The graph asymptotes towards large positive values i.e. in the same direction from either side of the asymptote. On the other hand, the linear factor (x+1) is raised to 1 i.e. odd power. Hence function value asymptotes in opposite directions.

Rational function

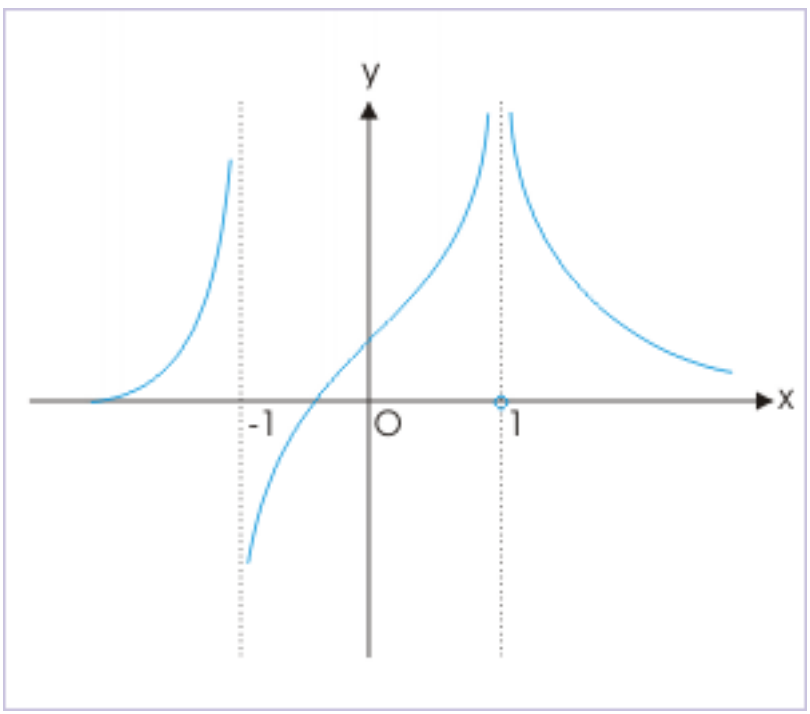

Figure 4: Asymptotes

2.3.2 Horizontal asymptote

Horizontal asymptote is a horizontal line including x-axis to which graph of function comes closer and closer but never touches. The difference between y-value and asymptote is infinitesimally small for large values of

x. An equation of horizontal asymptote has the form,

$$y = c$$

Existence of horizontal asymptote depends on the degree of polynomial in the numerator (n) and degree of polynomial in the denominator (m). There are following three cases :

1: If n>m, then there is no horizontal asymptote. However, if n=m+1, then there exists slant asymptote.

2: If n<m, then x-axis is horizontal asymptote.

3: If n=m, then there is horizontal asymptote exists. In this case, the equation of horizontal asymptote is :

$$y = \frac{\text{Coefficient of highest power term in numeratory}}{\text{Coefficient of highest power term in denominator}}$$

Exercise 2 ***(Solution on p. 20.)***

1. Find horizontal asymptote :

$$f(x) = \frac{2x^2 + x + 1}{x^2 + 16}$$

Exercise 3 ***(Solution on p. 20.)***

2. Find horizontal asymptote :

$$f(x) = \frac{-x(x+1)}{x^4 + 16}$$

Exercise 4 ***(Solution on p. 20.)***

3. Find horizontal asymptote :

$$f(x) = \frac{x\left(x^2 - 1\right)}{x + 2}$$

2.3.3 Slant asymptotes

Slant asymptote is a line that the graph approaches. This line is neither vertical nor horizontal. A rational function has a slant or oblique asymptote when order of numerator (n) is greater than order of denominator (m).

The equation of slant asymptote is obtained by dividing numerator polynomial by denominator polynomial. The quotient of division is equation of asymptote. Clearly, asymptote is a straight line. As such, quotient should be a linear expression. The requirement that asymptote is a straight line implies that the order of numerator polynomial is higher than order of denominator polynomial by 1 i.e. n=m+1.

In the nutshell, slant asymptote exists when n=m+1. The slant asymptote is obtained by dividing numerator and denominator. We neglect remainder. The equation of the slant asymptote is given by quotient equated to "y".

Exercise 5 ***(Solution on p. 20.)***

Find slant asymptote :

$$f(x) = \frac{x^2}{x + 1}$$

2.4 x-intercepts

The x-intercepts are also known as zeroes of function or real roots of corresponding equation when function is equated to zero. Since function is many-one, there can be more than one x-intercept. On graphs, x-intercepts are points on x-axis, where graph intersects it. Thus, x-intercepts are x-values where function value becomes zero.

$$f(x) = 0$$

In the case of rational function, x-intercepts exist when numerator turns zero. In other words, x-intercepts are x-values for which numerator of the function turns zero.

$$g(x) = \frac{f(x)}{h(x)} = 0$$

$$g(x) = 0 \quad \Rightarrow f(x) = 0$$

We determine x-intercepts by solving equation formed by equating function to zero. It is hepful to know that real polynomial of odd degree has a real root and, therefore, at least one x-intercept. In the case of rational function, the function is not defined for values of x when denominator turns zero. It means that there will be no x-intercept corresponding to linear factor which is common to denominator. Consider the function given here :

$$h(x) = \frac{(x-1)(x+2)}{(x-1)^2(x+1)}$$

Equating numerator to zero, we have :

$$\Rightarrow (x-1)(x+2) = 0$$

$$\Rightarrow x = 1, -2$$

But (x-1) is also linear factor in denominator. It means that point x=1 is a singularity. Hence, x-intercept is only x=-2 as function is not defined at x=1.

Exercise 6 ***(Solution on p. 20.)***

Find x-intercepts of reciprocal function :

$$f(x) = \frac{1}{x}$$

Exercise 7 ***(Solution on p. 20.)***

Find x-intercepts of function given by :

$$\frac{x^2 - 3x + 2}{x^2 - 2x - 3}$$

2.5 y-intercepts

This is function or y value when x is zero. Functions are many-one relation. Thus, there can be only one y-intercept. The y-intercept is calculated as :

$$y = f(0)$$

In the case of rational function, we can not determine y-intercept if the function is not defined at x=0.

Exercise 8 ***(Solution on p. 20.)***

Find y-intercepts of reciprocal function :

$$y = \frac{1}{x}$$

Exercise 9 ***(Solution on p. 20.)***

Find y-intercepts :

$$f(x) = 2x - 3$$

3 Real values of rational function

In order to find real values of rational function for real x, we rearrange the given rational function to form a quadratic equation in x. Let us consider a rational polynomial of degree 2 given by (we mean that the highest degree of polynomials involved in the ratio is 2) :

$$y = f(x) = \frac{a_1x^2 + b_1x + c_1}{a_2x^2 + b_2x + c_2}$$

Rearranging to form a quadratic equation in x, we have :

$$\Rightarrow y\left(a_2x^2 + b_2x + c_2\right) = a_1x^2 + b_1x + c_1$$

$$\Rightarrow (ya_2 - a_1)x^2 + (yb_2 - b_1)x + c_2y - c_1 = 0$$

For x to be real, D≥0. Hence,

$$\Rightarrow D = (yb_2 - b_1)^2 - 4x(ya_2 - a_1)(c_2y - c_1) \geq 0$$

We see that discriminant itself is a quadratic inequality. Depending on the nature of coefficient of "y^2 " in the quadratic equation and determinant of the corresponding quadratic equation, the inequality is solved for "y". This, in turn, allows us to determine the real interval(s) of y corresponding to real x.

We should clearly understand that these are real values of y corresponding to real x. This interval need not be the range of the function. Recall that range of a function contains values of y for values of x in the domain of function – not all real values of x. Now function is not defined for certain values of x, which are zeroes of denominator of the rational function. Hence domain is not the real number set R. This distinction should always be kept in mind.

Example 1

Problem : Find value of x for which given function has least value.

$$y = f(x) = \frac{x^2 - 6x + 5}{x^2 + 2x + 1}$$

Solution : Rearranging to form a quadratic equation in x, we have :

$$\Rightarrow yx^2 + 2yx + y = x^2 - 6x + 5$$

$$\Rightarrow (y-1)\,x^2 + 2\,(y+3)\,x + y - 5 = 0$$

For x real, D≥0.

$$\Rightarrow 4(y+3)^2 - 4\,(y-1)\,X\,(y-5) \geq 0$$

$$\Rightarrow \left(y^2 + 6y + 9\right) - \left(y^2 - 6y + 5\right) \geq 0$$

$$\Rightarrow 12y + 4 \geq 0$$

$$\Rightarrow y \geq -\frac{1}{3}$$

The real values of y, therefore, lies in the interval [-1/3, ∞). The least value of y in the interval of real values is -1/3. We should, however, check that value of y=-1/3 does not correspond to value of x which is not permitted. Here, the denominator polynomial is $x^2 + 2x + 1 = (x+1)^2$. Thus, $x \neq -1$. The domain of the function is R-{-1}.

Now, we calculate value of x corresponding to y as :

$$\Rightarrow -\frac{1}{3} = \frac{x^2 - 6x + 5}{x^2 + 2x + 1}$$

$$\Rightarrow x^2 - 4x + 4 = 0$$

$$\Rightarrow (x-2)^2 = 0$$

$$\Rightarrow x = 2$$

This point belongs to the domain of the function as it is different to excluded value. Hence, least value of y is -1/3.

Example 2

Problem : For what values of "a", the function given here assumes all real values for real x.

$$y = f\,(x) = \frac{ax^2 + 3x - 4}{3x - 4x^2 + a}$$

Solution : Rearranging to form a quadratic equation in x, we have :

$$\Rightarrow 3yx - 4yx^2 + ay = ax^2 + 3x - 4$$

$$\Rightarrow (a+4y)\,x^2 + 3\,(1-y)\,x - (ay+4) = 0$$

For x real, D≥0.

$$\Rightarrow 9(1-y)^2 + 4\,(a+4y)\,X\,(ay+4) \geq 0$$

$$\Rightarrow (9+16a)\,y^2 + \left(4a^2 + 46\right)y + (9+16a) \geq 0$$

This is a quadratic inequality in y. This inequality holds when coefficient of "y^2 " is positive and discriminant is less than equal to zero i.e. D'≤0.

$$\Rightarrow (9+16a) > 0$$

$$\Rightarrow a > -\frac{9}{16}$$

and

$$D\prime = \left(4a^2+46\right)^2 - 4(9+16a)^2 \leq 0$$

$$\Rightarrow \left(2a^2+23\right)^2 - (9+16a)^2 \leq 0$$

$$\Rightarrow \left(2a^2+23+9+16a\right)\left(2a^2+23-9-16a\right) \leq 0$$

$$\Rightarrow \left(2a^2+16a+32\right)\left(2a^2-16a+14\right) \leq 0$$

$$\Rightarrow \left(a^2+8a+16\right)\left(a^2-8a+7\right) \leq 0$$

$$\Rightarrow (a+4)^2\left(a^2-8a+7\right) \leq 0$$

$$\Rightarrow \left(a^2-8a+7\right) \leq 0$$

$$\Rightarrow \left(a^2-8a+7\right) \leq 0$$

The roots of corresponding quadratic equation in a is :

$$\Rightarrow a = 1, 7$$

Hence interval of a satisfying inequality is [1,7]

We have two intervals i.e. a>-9/16 and $\Rightarrow a \in [1,7]$ corresponding to two simultaneous conditions. Therefore, values of a is intersection of these two intervals :

$$\Rightarrow a \in [1,7]$$

4 Range of rational function

The set of real values of rational polynomial for real values of x need not be the range of the function. It is because rational function is not defined for zeroes of polynomial in denominator. In previous section, we evaluated values of function for real x. But, domain may not be the real number set, but subset of R, which excludes certain values of x. We need to exclude values of "y", which corresponds to values of x for which denominator becomes zero. This statement, however, is slightly confusing, because function is not defined for those values of x in the first place. How would we determine values of y corresponding to values of x for which function reduces to indeterminate form involving division by zero. We actually determine limiting values of function at these points and exclude those values of y from the real set of y, which is determined assuming x belonging to R.

There are certain cases in which denominator of the rational function can not become zero. Consider rational functions :

$$f(x) = \frac{2x^2 - x + 1}{x^2 + 1}$$

$$g(x) = \frac{x + 1}{2x^2 - x + 1}$$

$$h(x) = \frac{2x^2 - x + 1}{|x| + 1}$$

The denominators of all these functions can not be zero. Under this condition, domain of the function is real number set R.

Example 3
Problem : Find the range of function :

$$f(x) = \frac{x}{1 + x^2}$$

Solution : The denominator of the given rational function can not be zero. Hence, domain of function is real number set R. There is no exclusion point. Rearranging to form a quadratic equation in x, we have :

$$\Rightarrow y + yx^2 = x$$

$$\Rightarrow yx^2 - x + y = 0$$

We should analyze for coefficient of "x^2" in the quadratic equation. For quadratic equation, coefficient of "x^2" can not be zero i.e. y $\neq$ 0. For real x, y $\neq$ 0 and D$\geq$0 :

$$D = (-1)^2 - 4XyXy = 1 - 4y^2 \geq 0$$

$$\Rightarrow y^2 \geq \frac{1}{4}$$

$$\Rightarrow y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

What if y=0? Putting this value in the quadratic equation, we have :

$$\Rightarrow 0 - x + 0 = 0$$

$$\Rightarrow x = 0$$

This is included in the domain. Hence, y=0 is included in the range. The range of the rational function, therefore, remains unaffected :

$$\Rightarrow y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

Example 4
Problem : Find the range of the function :

$$y = f(x) = \frac{x^2 - 5x + 4}{x^2 - 3x + 2}$$

Solution : We see that discrimanants of numerator and denominator polynomials are positive. On factorizing,

$$\Rightarrow y = \frac{x^2 - 5x + 4}{x^2 - 3x + 2} = \frac{(x-1)(x-4)}{(x-1)(x-2)}$$

Clearly, rational function is not defined for x=1 and x=2. Domain of the function is R- {1,2). For the sake of determining range, the limiting values of function for these values of x are obtained by canceling (x-1) from numerator and denominator :

$$\Rightarrow y = \frac{(x-4)}{(x-2)}$$

For x=1, y = 3. For x=2, however, the function value is indeterminate. In totality, we need to exclude y=3 from the interval of real values of y. Now, in order to determine real values of y, we rearrange the given function to form a quadratic equation in x :

$$\Rightarrow yx^2 - 3yx + 2y = x^2 - 5x + 4$$

$$\Rightarrow (y-1)x^2 + (5-3y)x + 2y - 4 = 0$$

We should analyze for coefficient of "x^2 " in the quadratic equation. For quadratic equation, coefficient of "x^2" can not be zero i.e. y-1 $\neq$ 0. For real x, y-1 $\neq$ 0 and D$\geq$0.

For y-1 = 0, y = 1. Putting this value in the quadratic equation,

$$\Rightarrow 0 + (5-3)x + 2 - 4 = 0$$

$$\Rightarrow x = 1$$

We see that x=1 is not part of domain. This is actually the value which reduces denominator to zero. Hence, we should exclude y = 1 from the real values of y. Now for D$\geq$0,

$$D = (5-3y)^2 - 4(y-1)(2y-4) \geq 0$$

$$\Rightarrow 25 + 9y^2 - 30y - 4\{2y^2 - 6y + 4\} \geq 0$$

$$\Rightarrow 25 + 9y^2 - 30y - 4\{2y^2 - 6y + 4\} \geq 0$$

The coefficient of

$$y^2$$

is positive. The discriminant is 0. Clearly, following sign rule, f(x) $\geq$0 for all real values of y. Hence, real values of y are real number set R. However, we need to exclude y = {1,3) as discussed above. Therefore, range of given function is R-{1,3}.

Alternative

Once, exception points are noted, we can evaluate "y" from the reduced form :

$$\Rightarrow y = \frac{(x-4)}{(x-2)}$$

Solving,

$$\Rightarrow x = \frac{2y-4}{(y-1)}$$

Clearly, y#1. But we have seen that y#3 as well. Hence, range of rational function is R-{1,3}.

5 Graph of rational function

We know that rational function is a composition of two functions in the following form,

$$f(x) = \frac{p(x)}{q(x)}$$

where q(x) $\neq$ 0. If q(x) = 0, then the ratio has the form " $x/0$ ", which is not defined.

For plotting, let us consider a simple rational function given by, $f(x) = 1/x$. This function is known as reciprocal function. It is not defined for x = 0. In order to plot the function, we calculate few initial values as :

$$For \quad x = -1, \quad y = -1$$

$$For \quad x = -2, \quad y = -0.5$$

$$For \quad x = -3, \quad y = -0.33$$

$$For \quad x = 0, \quad \text{y is not defined}$$

$$For \quad x = 1, \quad y = 1$$

$$For \quad x = 2, \quad y = 0.5$$

$$For \quad x = 3, \quad y = 0.33$$

The graph of the function is shown here :

Rational function

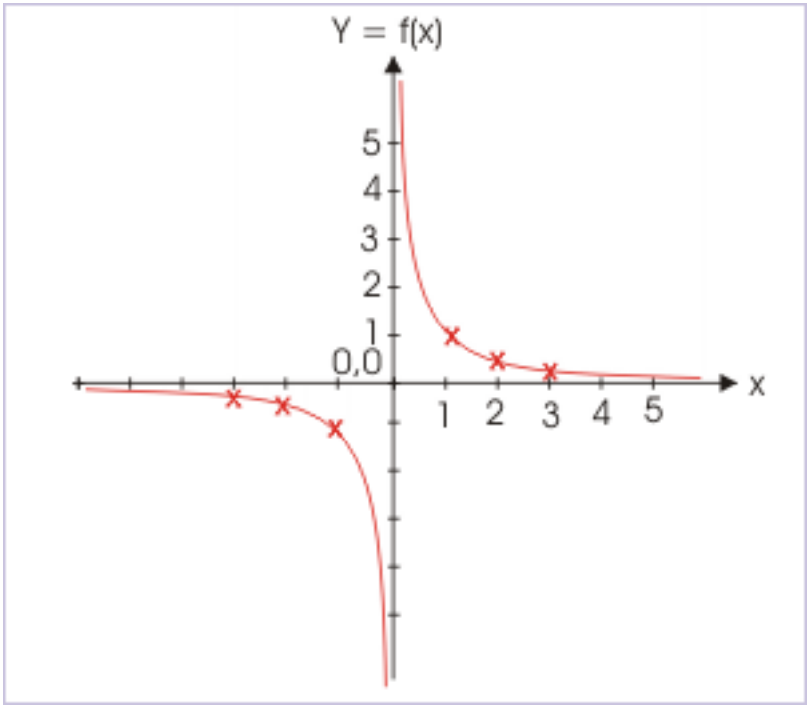

Figure 5: This plot is not defined at x = 0.

This plot is not defined at x = 0. The domain of the given function, therefore, is real numbers, "R" except zero. Also,

$$\Rightarrow x = \frac{1}{y}$$

This means that function value can not be zero. Hence, range of the function is also real numbers, "R" except zero.

$$Domain = R - \{0\}$$

$$Range = R - \{0\}$$

Example 5

Problem : Draw the graph of rational function given by :

$$f(x) = \frac{x^2 - 1}{x - 1}$$

Discuss the nature of graph and also determine domain and range of the given function.

Solution : The form of the given function is that of rational function. We observe that the function is not defined for "x = 1" as function has the form " $x/0$ ", which is undefined. The domain

of the given function, therefore, is “R” except “1”. It should be noted that while interpreting domain or range we should not cancel out common terms in the numerator and denominator.

For other values of “x”, the value of the function is given by the reduced expression :

$$f(x) = \frac{x^2 - 1}{x - 1} = x + 1$$

Clearly, if the given function were valid for x =1, then y = x+1 = 1 + 1 = 2. Thus, function f(x) can take any real value except “2”. Hence, range of the function is "R" except "2". The domain and range of the given function are :

$$Domain = R - \{1\}$$

$$Range = R - \{2\}$$

In order to plot the function, we calculate few initial values as :

$$For \quad x = -3, \quad y = -2$$

$$For \quad x = -2, \quad y = -1$$

$$For \quad x = -1, \quad y = 0$$

$$For \quad x = 0, \quad y = 1$$

$$For \quad x = 1, \quad \text{y is not defined}$$

$$For \quad x = 2, \quad y = 3$$

$$For \quad x = 3, \quad y = 4$$

The graph of the function is shown here :

Rational function

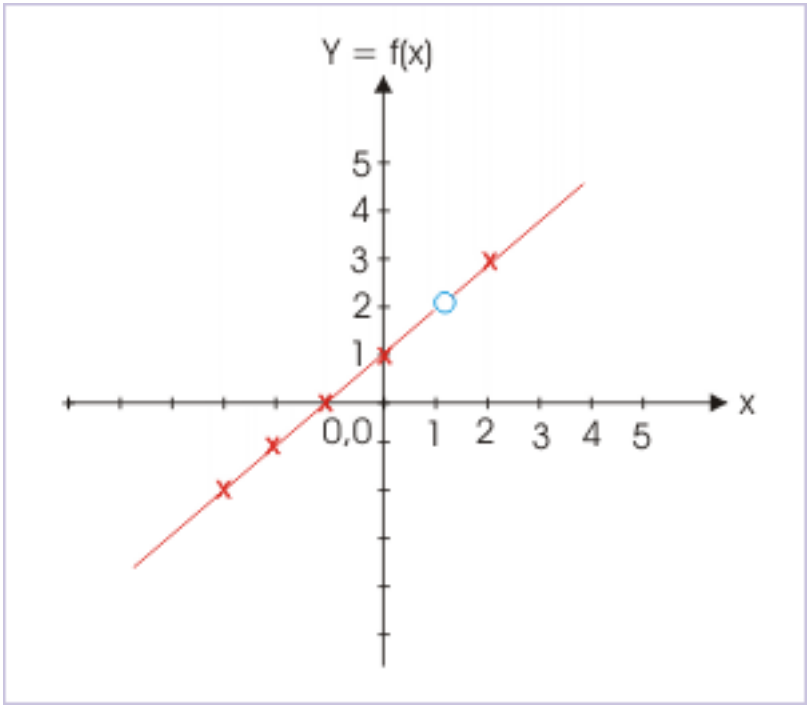

Figure 6: The plot is not defined at x = 1.

The plot is not defined at x = 1. There is a break at x = 1.

5.1 Nature of graph

Here, we consider graphs of rational functions of type :

$$y = \frac{1}{x}, \quad \frac{1}{x^3}, \quad \frac{1}{x^5}, \ldots\ldots\ldots\ldots$$

The nature of graph of these rational function of type $y = \frac{1}{x^n}$, where n is an odd integer such that n$\geq$ 1, is similar to graph of y=1/x as shown in the figure. The graph is that of rectangular hyperbola.

Rational function

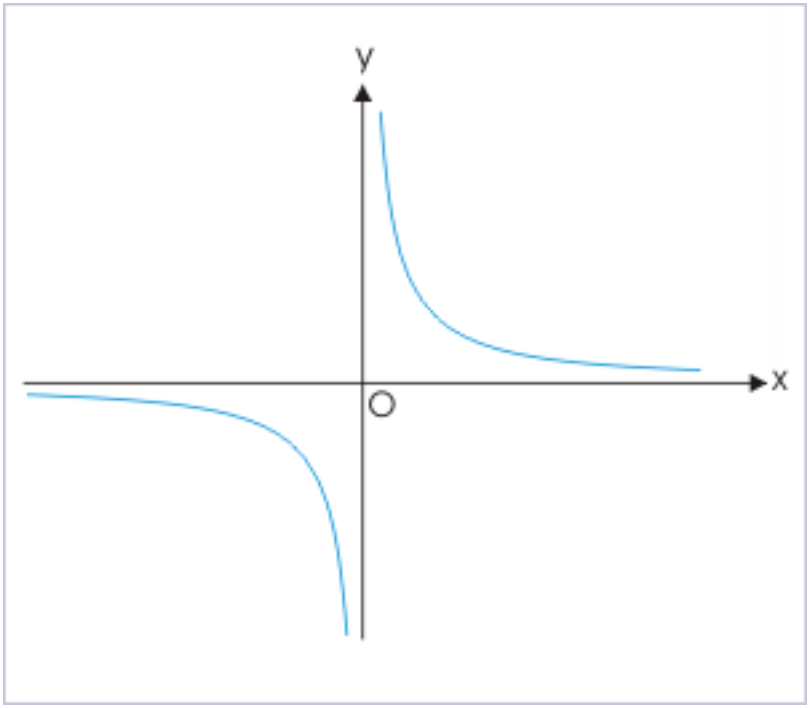

Figure 7: Graph of rational function.

We need to emphasize that the graph generalizes the nature and is helpful to estimate domain and range of functions. We need to graph individual function if required. The nature of graph of function type $y = \frac{1}{x^n}$, where n is an even integer such that n≥2 is shown in the figure below :

$$y = \frac{1}{x^2}, \quad \frac{1}{x^4}, \quad \frac{1}{x^4}, \ldots\ldots\ldots\ldots$$

Rational function

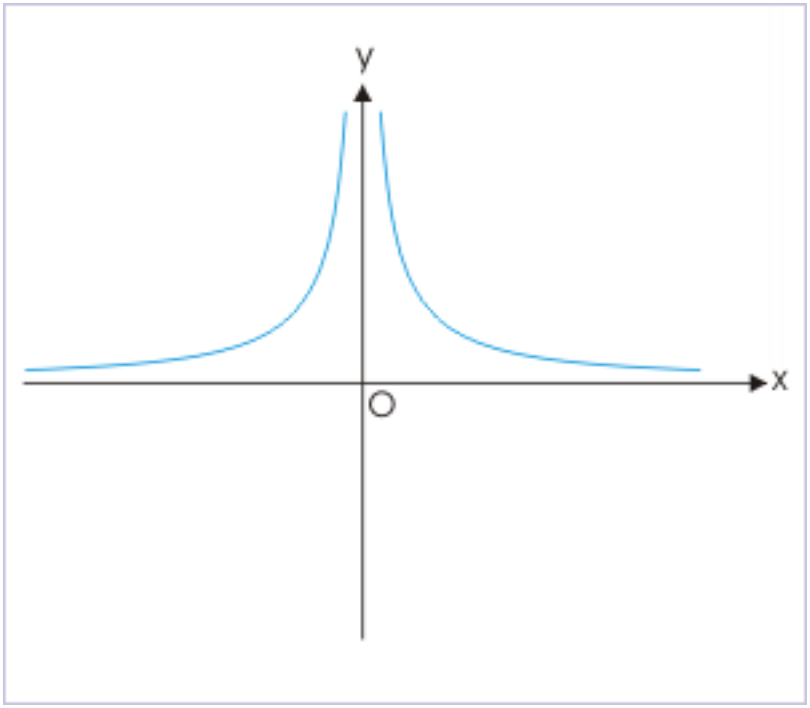

Figure 8: Graph of rational function.

Solutions to Exercises in this Module

Solution to Exercise (p. 2)
Factorizing into linear factors, we have :

$$\frac{x^2-3x+2}{x^2-2x-3}=\frac{(x-1)(x-2)}{(x+1)(x-3)}$$

Equating denominator to zero, we have :

$$\Rightarrow (x+1)(x-3)=0$$

$$\Rightarrow x=-1 \quad \text{or} \quad 3$$

Thus, singularities are -1 and 3.

Solution to Exercise (p. 7)
Here, coefficient of highest power term is 2 in numerator and 1 in denominator. Hence, horizontal asymptote is given by :

$$y=\frac{2}{1}=2$$

Solution to Exercise (p. 7)
The order of highest power term is 2 in numerator and 4 in denominator. Thus, n<m. Hence, x-axis is horizontal asymptote.

$$y=0$$

Solution to Exercise (p. 7)
The order of highest power term is 3 in numerator and 1 in denominator. Here, n>m. Hence, there is no horizontal asymptote.

Solution to Exercise (p. 7)
Division here yields quotient as "x-1". Hence, equation of slant asymptote is :

$$y=x-1$$

Solution to Exercise (p. 8)
Here numerator is 1 and can not be zero. Thus, reciprocal function does not have x-intercepts.

Solution to Exercise (p. 8)
Factorizing, we have :

$$\frac{x^2-3x+2}{x^2-2x-3}=\frac{(x-1)(x-2)}{(x+1)(x-3)}$$

Equating numerator to zero, we have :

$$\Rightarrow (x-1)(x-2)=0$$

$$\Rightarrow x=1 \quad \text{or} \quad 2$$

Thus, x-intercepts are 1 and 2.

Solution to Exercise (p. 9)
The equation can not be solved for x=0. Thus, reciprocal function does not have y-intercepts.

Solution to Exercise (p. 9)
Putting x=0,

$$f(0) = -3$$