Value of a function*

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The value of a function at "x = a" is denoted as "f(a)". The working rule for finding value of a function is to replace independent variable "x" by "a".

1 Polynomial and rational functions

1.1

Problem 1: Find "f(y)", if

$$y = f(x) = \frac{1-x}{1+x}$$

Solution :

Statement of the problem : The given function is a rational function. We have to evaluate the function when independent variable is function itself.

We need to replace "x" by "y".

$$\Rightarrow f(y) = \frac{1-y}{1+y} = \frac{1-\frac{1-x}{1+x}}{1+\frac{1-x}{1+x}}$$

$$\Rightarrow f(y) = \frac{1+x-1+x}{1+x+1-x} = \frac{2x}{2} = x$$

1.2

Problem 2: Find "f(x)", if

$$f(x-1) = \left(x^2-1\right)$$

Solution :

Statement of the problem : The given function is a polynomial function with a polynomial as its argument. We have to evaluate the function for independent variable "x".

We need to replace "x-1" by "x" in the given equation to find "f(x)". The right hand side expression, however, does not contain term "x-1". We, therefore, need to find the term, which will replace "x". Clearly if "x" replaces "x-1", then "x+1" will replace "x-1+1 = x"

Thus, we need to replace "x" by "x+1".

$$\Rightarrow f(x+1-1) = (x+1)^2 - 1 = x^2 + 2x + 1 - 1 = x^2 + 2x$$

*Version 1.4: Oct 27, 2008 1:02 am -0500

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1.3

Problem 3: If $\{f(x)\} = x + \frac{1}{x}$, then prove that :

$$\Rightarrow \{f(x)\}^3 = f\left(x^3\right) + 3f\left(\frac{1}{x}\right)$$

Solution :

Statement of the problem : The function is a polynomial function. We have to evaluate cube of the function, which involves evaluation of function for arguments, which are independent variable, raised to certain integral powers.

The cube of given function is :

$$\Rightarrow \{f(x)\}^3 = \left(x + \frac{1}{x}\right)^3 = x^3 + 1/x^3 + 3x^2 X \frac{1}{x} + 3x X \frac{1}{x^2}$$

$$\Rightarrow \{f(x)\}^3 = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right)$$

Now, $f\left(x^3\right)$ is :

$$f\left(x^3\right) = x^3 + \frac{1}{x^3}$$

Hence,

$$\{f(x)\}^3 = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) = f\left(x^3\right) + 3f(x)$$

But, we see that :

$$\Rightarrow f\left(\frac{1}{x}\right) = \frac{1}{x} + x = f(x)$$

Hence,

$$\{f(x)\}^3 = f\left(x^3\right) + 3f(x) = f\left(x^3\right) + 3f\left(\frac{1}{x}\right)$$

1.4

Problem 4: If

$$f(x) = \frac{(1+x)}{(1-x)}$$

Then, find

$$\frac{f(x)\, f\left(x^2\right)}{1 + \{f(x)\}^2}$$

Solution :

Statement of the problem : The given function is rational function. We have to find the expression which involves (i) function, (ii) function with argument as squared independent variable and (iii) square of the function.

We need to substitute for various terms in the given expression :

$$\Rightarrow \frac{f(x)\,f(x^2)}{1+\{f(x)\}^2} = \frac{\left(\frac{1+x}{1-x}\right) X \left(\frac{1+x^2}{1-x^2}\right)}{1+\left(\frac{1+x}{1-x}\right)^2}$$

$$= \frac{\frac{(1+x)(1+x^2)}{(1-x)(1-x^2)}}{\frac{(1-x)^2+(1+x)^2}{(1-x)^2}}$$

$$\frac{\frac{(1+x^2)}{(1-x)^2}}{\frac{2(1+x^2)}{(1-x)^2}} = \frac{(1+x^2)}{2(1+x^2)} = \frac{1}{2}$$

2 Modulus functions

2.1

Problem 5: If

$$f(x) = \frac{|x|}{x};\, x \neq 0$$

Then, evaluate

$$|f(a) - f(-a)|$$

Solution :

Statement of the problem : Function, f(x), involves modulus and is in rational form. The value of this function, in turn, forms the part of a expression to be evaluated. We have to find the value of expression.

We first evaluate the expression without modulus sign :

$$f(a) - f(-a) = \frac{|a|}{a} - \frac{|-a|}{-a} = \frac{|a|}{a} + \frac{|a|}{a} = \frac{2|a|}{a}; \quad a \neq 0$$

But, we know that $|a| = \pm a$

$$\Rightarrow f(a) - f(-a) = \frac{2X \pm a}{a} = \pm 2$$

Taking modulus of the expression,

$$\Rightarrow |f(a) - f(-a)| = 2; \quad a \neq 0$$

Note that we need to keep the condition for which the given expression is evaluated.

3 Logarithmic functions

3.1

Problem 6: Find $f\left(\frac{2x}{1+x^2}\right)$, if

$$f(x) = \log_e\left(\frac{1+x}{1-x}\right)$$

Solution :

Statement of the problem : The given function is transcendental logarithmic function. We have to evaluate the function for an argument (input to function), which is itself a rational function in independent variable, "x".

We need to replace "x" by "$2x/1+x^2$".

$$\Rightarrow f\left(\frac{2x}{1+x^2}\right) = \log_e\left(\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right) = \log_e\left(\frac{1+x^2+2x}{1+x^2-2x}\right)$$

$$\Rightarrow f\left(\frac{2x}{1+x^2}\right) = \log_e\left(\frac{1+x}{1-x}\right)^2 = 2\log_e\left(\frac{1+x}{1-x}\right) = 2f\left(x\right)$$

4 Trigonometric functions

4.1

Problem 7: Find $f\left(\pi/4\right)$, if

$$f\left(x\right) = \frac{2\cot x}{1+\cot^2 x}$$

Solution :

Statement of the problem : The given function is a rational function with trigonometric function as independent variable. We have to find the value of function for a particular angle.

We need to replace "x" by "$\pi/4$ ".

$$\Rightarrow f\left(\pi/4\right) = \frac{2\cot\left(\pi/4\right)}{1+\cot^2\left(\pi/4\right)}$$

As $\cot\left(\pi/4\right) = 1$,

$$\Rightarrow f\left(\frac{\pi}{4}\right) = \frac{2X1}{1+1^2} = 1$$

4.2

Problem 8: Find $f\left(\tan\theta\right)$, if

$$f\left(x\right) = \frac{2x}{1+x^2}$$

Solution :

Statement of the problem : The given function is a rational function. We have to evaluate the function for a value, which is itself a trigonometric function.

We need to replace "x" by "$\tan\theta$".

$$\Rightarrow f\left(\tan\theta\right) = \frac{2\left(\tan\theta\right)}{1+\tan^2\theta} = \sin 2\theta$$

4.3

Problem 9: If $f\left(x\right)=\cos\{\log\left(x\right)\}$, then prove that :

$$f\left(xy\right)+f\left(\frac{x}{y}\right)=2f\left(x\right)f\left(y\right)$$

Solution :

Statement of the problem : The given function, f(x) is a trigonometric function, whose input is a logarithmic function. We have to evaluate LHS of the given equation to equate the same to RHS.

Here, we evaluate each term of the left hand side of the equation separately and then combine the result.

$$\Rightarrow f\left(xy\right)=\cos\{\log_e\left(xy\right)\}=\cos\left(\log_e x+\log_e y\right)$$

$$f\left(\frac{x}{y}\right)=\cos\{\log_e\left(\frac{x}{y}\right)\}=\cos\left(\log_e x-\log_e y\right)$$

Substituting in the LHS expression, we have :

$$\Rightarrow \{f\left(xy\right)+f\left(\frac{x}{y}\right)\}=\cos\left(\log_e x+\log_e y\right)+\cos\left(\log_e x-\log_e y\right)$$

We know that :

$$\cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

Hence,

$$\Rightarrow \{f\left(xy\right)+f\left(\frac{x}{y}\right)\}=2\cos\left(\frac{\log_e x+\log_e y+\log_e x-\log_e y}{2}\right)\cos\left(\frac{\log_e x+\log_e y-\log_e x+\log_e y}{2}\right)$$

$$\Rightarrow \{f\left(xy\right)+f\left(\frac{x}{y}\right)\}=2\cos\left(\log_e x\right)\cos\left(\log_e y\right)$$

$$\{f\left(xy\right)+f\left(\frac{x}{y}\right)\}=2\cos\left(\log_e x\right)\cos\left(\log_e y\right)=2f\left(x\right)f\left(y\right)$$

5 Acknowledgment

Author wishes to thank Mr Jay Sicard, Pasadena TX for suggesting valuable typographical correction on the topic.