Domain and range of exponential and logarithmic function*

Sunil Kumar Singh

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Working rules : We shall be using following definitions/results for solving problems in this module :

- $y = \log_a x, \quad \text{where} \quad a > 0, a \neq 1, x > 0, y \in R$
- $y = \log_a x \Leftrightarrow x = a^y$
- If $\log_a x \geq y, \quad \text{then} \quad x \geq a^y, \quad \text{if} \quad a > 1$
- If $\log_a x \geq y, \quad \text{then} \quad x \leq a^y, \quad \text{if} \quad a < 1$

1 Domain of different logarithmic functions

Example 1

Problem : Find the domain of the function given by (Be aware that "x" appears as base of given logrithmic function):

$$f(x) = \log_x 2$$

Solution : By definition of logarithmic function, we know that base of logarithmic function is a positive number excluding x =1.

$$x > 0, \quad x \neq 1$$

Hence, domain of the given function is :

Domain of the function

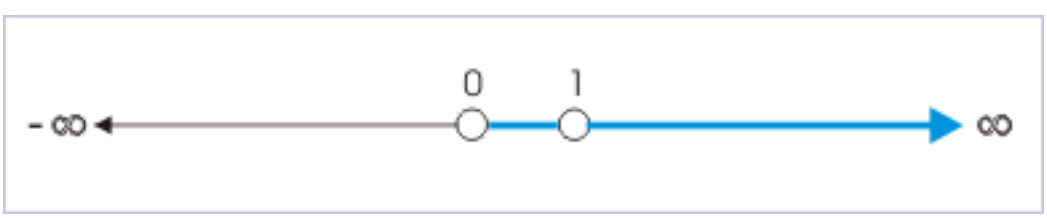

Figure 1: Thick line represents domain of the given function.

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$$\text{Domain} = (0, \infty) - \{1\}$$

or,

$$\text{Domain} = (0, 1) \cup \{1, \infty\}$$

Example 2
Problem : Find the domain of the function given by :

$$f(x) = \log_{10} \frac{x^2 - 5x + 6}{x^2 + 5x + 9}$$

Solution : The argument (input to the function) of logarithmic function is a rational function. We need to find values of "x" such that the argument of the function evaluates to a positive number. Hence,

$$\Rightarrow \frac{x^2 - 5x + 6}{x^2 + 5x + 9} > 0$$

In this case, we can not apply sign scheme for the rational function as a whole. Reason is that the quadratic equation in the denominator has no real roots and as such can not be factorized in linear factors. We see that discreminant,"D", of the quadratic equation in the denominator, is negative :

$$\Rightarrow D = b^2 - 4ac = 5^2 - 4X1X9 = 25 - 36 = -11$$

The quadratic expression in denominator is positive for all value of x as coefficient of squared term is positive. Clearly, sign of rational function is same as that of quadratic expression in the numerator. The coefficient of squared term of the numerator "x^2", is positive for all values of "x". The quadratic expression in the numerator evaluates to positive for intervals beyond root values. The roots of the corresponding equal equation is :

$$\Rightarrow x^2 - 2x - 3x + 6 = 0 \quad \Rightarrow x(x-2) - 3(x-2) = 0 \quad \Rightarrow (x-2)(x-3) = 0$$

Domain of the function

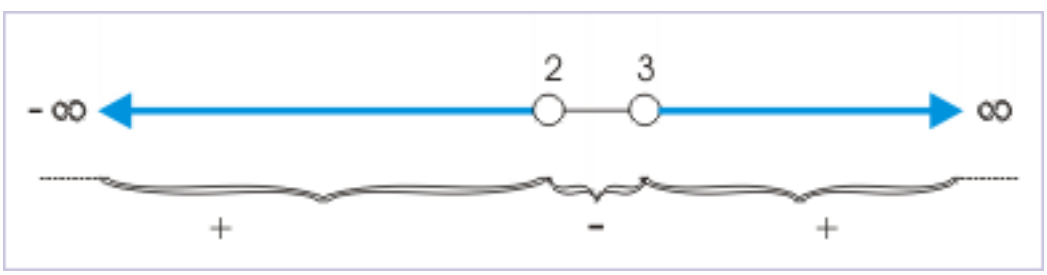

Figure 2: Thick line represents domain of the given function.

$$x < 2 \quad \text{or} \quad x > 3$$

$$\text{Domain} = (-\infty, 2) \quad \cup \quad (3, \infty)$$

Example 3

Problem : Find the domain of the function given by :

$$f(x) = \sqrt{\log_{10} \frac{6x - x^2}{8}}$$

Solution : The function is a square root of a logarithmic function. On the other hand argument of logarithmic function is a rational function. In order to find the domain of the given function, we first determine what values of "x" are valid for logarithmic function. Then, we apply the condition that expression within square root should be non-negative number. Domain of given function is intersection of intervals of x obtained for each of these conditions. Now, we know that argument (input to function) of logarithmic function is a positive number. This implies that we need to find the interval of "x" for which,

$$\Rightarrow \frac{6x - x^2}{8} > 0$$

$$\Rightarrow 6x - x^2 > 0$$

In above step, we should emphasize here that we multiply "8" and "0" and retain the inequality sign because 8>0. Now, we multiply the inequality by "-1". Therefore, inequality sign is reversed.

$$\Rightarrow x^2 - 6x < 0$$

Here, roots of corresponding quadratic equation " $x^2 - 6x$" is x = 0, 6. It means that middle interval between "0 and 6" is negative as coefficient of "x^2" is positive i.e. 6>0. Hence, interval satisfying the inequality is :

Domain of the function

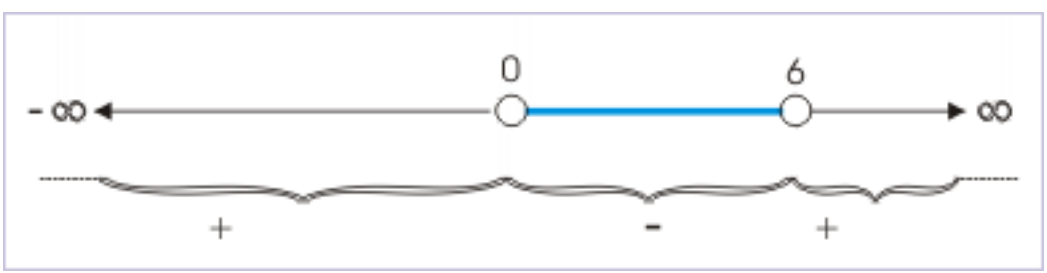

Figure 3: Thick line represents domain of the given function.

$$0 < x < 6$$

Now, we interpret second condition according to which the whole logarithmic expression within the square root should be a non-negative number.

$$\Rightarrow \log_{10} \frac{6x - x^2}{8} \geq 0$$

We use the fact that if $\log_a x \geq y$, then $x \geq a^y$ for $a > 1$. This gives us the inequality as given here,

$$\Rightarrow \frac{6x - x^2}{8} \geq 10^0$$

$$\Rightarrow \frac{6x - x^2}{8} \geq 1$$

$$\Rightarrow 6x - x^2 \geq 8 \quad \Rightarrow 6x - x^2 - 8 \geq 0$$

$$\Rightarrow x^2 - 6x + 8 \leq 0 \quad \Rightarrow x^2 - 2x - 4x + 8 \leq 0$$

$$\Rightarrow x(x-2) - 4(x-2) \leq 0 \quad \Rightarrow (x-2)(x-4) \leq 0$$

Clearly, "2" and "4" are the roots of the corresponding quadratic equation. Following sign scheme, we pick middle negative interval :

Domain of the function

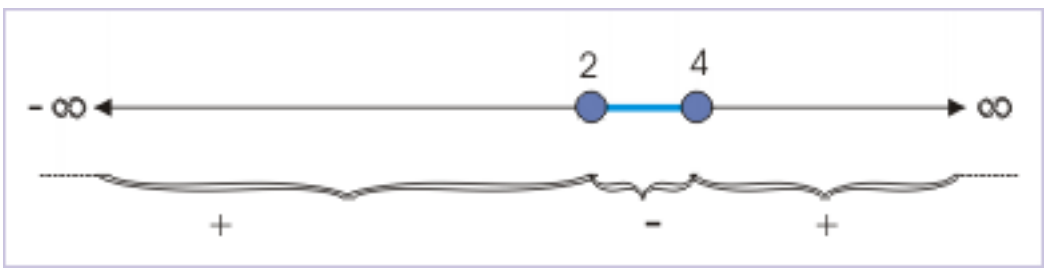

Figure 4: Thick line represents domain of the given function.

$$2 \leq x \leq 4$$

Now, the interval of "x" valid for real values of "f(x)" is the one which satisfies both conditions simultaneously i.e. the interval common to two intervals determined. Hence,

Domain of the function

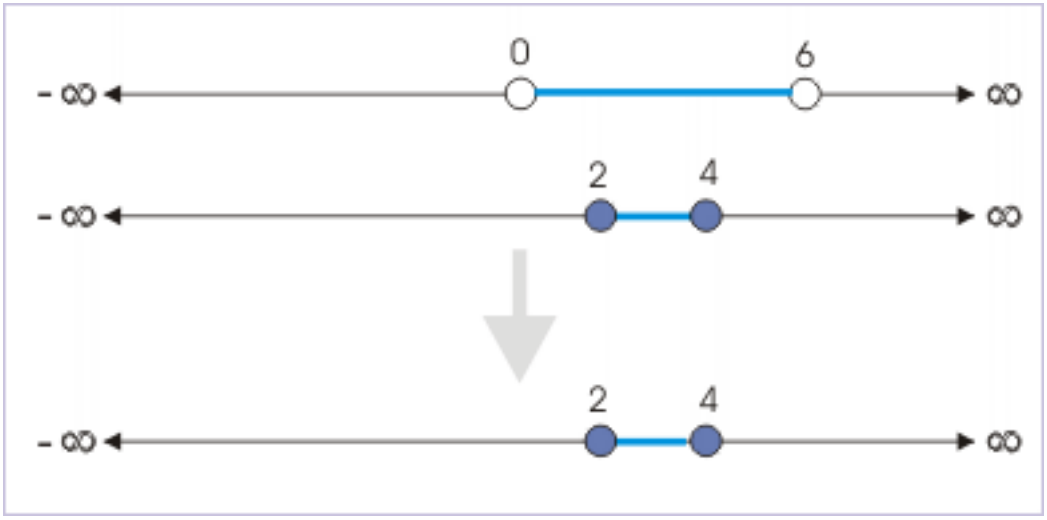

Figure 5: Thick line represents domain of the given function.

$$\text{Domain} = 0 < x < 6 \quad \cap \quad 2 \leq x \leq 4$$

$$\text{Domain} = 2 \leq x \leq 4 = [2, 4]$$

Example 4
Problem : Find the domain of the function given by :

$$f(x) = \sqrt{\{(\log_{0.2} x)^3 + \log_{0.2} x^3 X \log_{0.2} 0.0016x + 36\}}$$

Solution : The function is square root of an expression, consisting logarithmic functions. Here, we first need to simplify expression, using logarithmic identities, before attempting to find domain of the function. Let us first simplify the middle term of the given expression, using logarithmic identities :

$$\log_{0.2} x^3 X \log_{0.2} 0.0016x = 3\log_{0.2} x X \log_{0.2} 0.2^4 x$$

$$\Rightarrow \log_{0.2} x^3 X \log_{0.2} 0.0016x = 3\log_{0.2} x X \left(4\log_{0.2} 0.2 + \log_{0.2} x\right)$$

We observe that all logarithmic functions have the base of "0.2". Let us consider that $z = \log_{0.2} x$, then logarithmic expression within square root is :

$$\Rightarrow z^3 + 3z(4 + z) + 36 = z^3 + 3z^2 + 12z + 36 = z^2(z + 3) + 12(z + 3)$$

$$\Rightarrow z^3 + 3z(4 + z) + 36 = \left(z^2 + 12\right)(z + 3)$$

Now, this expression is non-negative for square root to be real. Hence,

$$\Rightarrow \left(z^2+12\right)(z+3) \geq 0$$

But, we see that z^2+12 is a positive number as term z^2 is positive. It means that :

$$\Rightarrow (z+3) \geq 0 \quad \Rightarrow \text{loglog}_{0.2}x \geq -3 \quad \Rightarrow x \leq 0.2^{-3} \quad \Rightarrow x \leq \frac{1}{0.008} \quad \Rightarrow x \leq 125$$

Note that we have reversed the inequality as the base is 0.2, which is less than 1. Further, we have substituted as :

$$z = \log_{0.2}x$$

This logarithmic function is valid by definition for all positive value of "x". Now, the domain of given function is the intersection of two intervals as shown in the figure.

Domain of the function

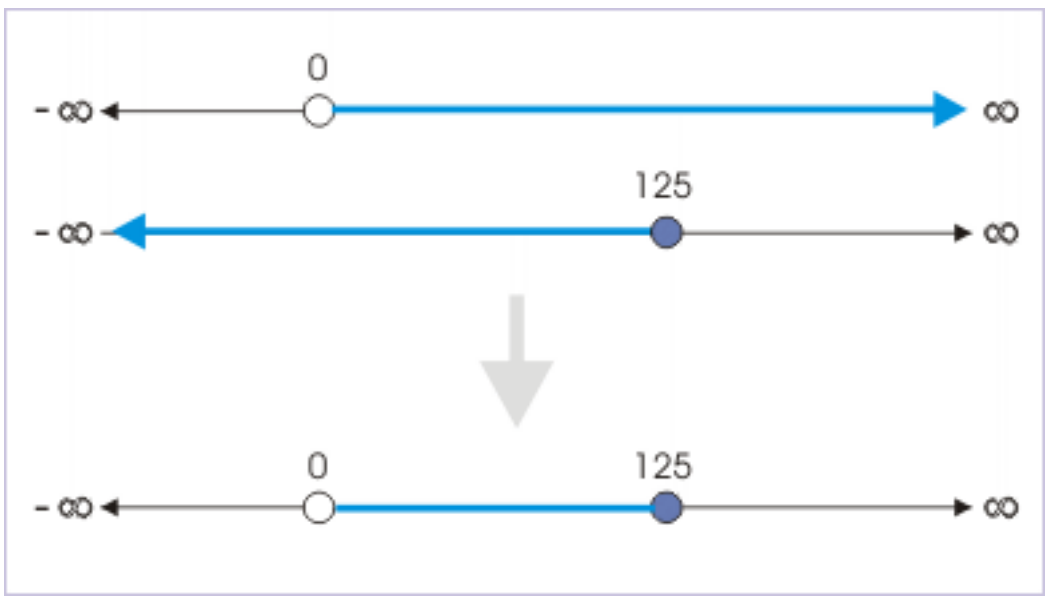

Figure 6: Thick line represents domain of the given function.

$$\text{Domain} = (0, 125]$$

2 Range of logarithmic function

Example 5
Problem : Find range of the function :

$$f(x) = \frac{e^x - e^{-|x|}}{e^x + e|x|}$$

Solution : We observe that for x≤0,

$$f(x) = 0$$

For x>0

$$\Rightarrow y = f(x) = \frac{e^x - e^{-|x|}}{e^x + e|x|} = \frac{e^{2x} - 1}{2e^{2x}}$$

$$\Rightarrow yX2e^{2x} = e^{2x} - 1 \quad \Rightarrow (1 - 2y)\, e^{2x} = 1$$

$$\Rightarrow e^{2x} = \frac{1}{1 - 2y}$$

We can see that $e^{2x} \geq 1$ for all x. Hence,

$$\Rightarrow \frac{1}{1 - 2y} \geq 1 \quad \Rightarrow \frac{1}{1 - 2y} - 1 \geq 0 \quad \Rightarrow \frac{1 - 1 + 2y}{1 - 2y} \geq 0$$

$$\Rightarrow \frac{2y}{1 - 2y} \geq 0$$

$$\Rightarrow \frac{2y}{2y - 1} \leq 0$$

Here, critical points are 0,1. Thus, range of the given function is :

$$\text{Range} = \left[0, \frac{1}{2}\right)$$

3 Exercise

Exercise 1 *(Solution on p. 9.)*

Find the domain of the function given by :

$$f(x) = 2^{\sin^{-1}(x)}$$

Exercise 2 *(Solution on p. 9.)*

Find the domain of the function given by :

$$f(x) = \log_{10}\{\sqrt{(8 - x)} + \sqrt{(x - 2)}\}$$

Exercise 3 *(Solution on p. 9.)*

Find the domain of the function :

$$f(x) = \log_{10}\{1 - \log\left(x^2 - 3x + 12\right)\}$$

Exercise 4 *(Solution on p. 10.)*

Problem 3 : Find the domain of the function given by :

$$f(x) = \log_2 \log_3 \log_4 x$$

Exercise 5 *(Solution on p. 11.)*

Find the range of the function :

$$f(x) = \log_{10}\left(x^2 - 3x + 4\right)$$

Exercise 6 ***(Solution on p. 12.)***

Find domain and range if

$$e^x - e^{f(x)} = e$$

Solutions to Exercises in this Module

Solution to Exercise (p. 7)

The exponent of the exponential function is inverse trigonometric function. Exponential function is real for all real values of exponent. We see here that given function is real for the values of "x" corresponding to which arcsine function is real. Now, domain of arcsine function is [-1,1]. This is the interval of "x" for which arcsine is real. Hence, domain of the given function, "f(x)" is :

$$\text{Domain} = [-1, 1]$$

Solution to Exercise (p. 7)

The argument (input to the function) of logarithmic function is addition of two square roots. We need to find values of "x" such that the argument of the logarithmic function evaluates to a positive number. An unsigned square root is a positive number by definition. It can not be negative. Symbolically, $\sqrt{\text{x}}$ is a positive number. Clearly, each of the square roots is a positive number. Hence, their addition is also a positive number. Thus, we see that the requirement of the argument of a logarithmic function being a positive number, is automatically fulfilled by virtue of the property of an unsigned square root.

We, therefore, only need to evaluate "x" for which each of the square roots is real. In other words, the expressions in each of the square roots is a non-negative integer.

$$8 - x \geq 0 \quad \Rightarrow x - 8 \leq 0 \quad \Rightarrow x \leq 8$$

$$x - 2 \geq 0 \quad \Rightarrow x \geq 2$$

The two square root functions are added to form the argument of logarithmic function. We know that domain of function resulting from addition is intersection of domains of individual square root function. Hence,

$$\text{Domain} = [2, 8]$$

Solution to Exercise (p. 7)

Hints : There are two logarithmic functions composing the given function. Let us call them outer and inner. For outer logarithmic function,

$$\Rightarrow 1 - \log\left(x^2 - 3x + 12\right) > 0$$

$$\Rightarrow \log\left(x^2 - 3x + 12\right) < 1$$

$$\Rightarrow \log_{10}\left(x^2 - 3x + 12\right) < \log_{10} 10$$

$$\Rightarrow x^2 - 3x + 12 < 10$$

$$\Rightarrow x^2 - 3x + 2 < 0$$

$$\Rightarrow (x - 1)(x - 2) < 0$$

$$\Rightarrow x \in (1, 2)$$

For inner logarithmic function,

$$x^2 - 3x + 12 > 0$$

Here, coefficient of squared term is positive and and D<0. Hence, this inequality is true for all real x i.e. x [U+F0CE] R. Now, domain of given function is intersection of two intervals.

$$\text{Domain} = (1, 2)$$

Solution to Exercise (p. 7)

The function is formed by nesting three logarithmic functions. Further base of logarithmic functions are different. For determining domain we (i) find value of "x" for which "$\log_4 x$" is real (ii) find range of "$\log_4 x$" for which "$\log_3 (\log_4 x)$" is real and (iii) find range of "$\log_3 (\log_4 x)$" for which f(x) is real.

For "$\log_4 x$" to be real, x is a positive number. It means,

$$x > 0$$

For "$\log_3 (\log_4 x)$" to be real, "$\log_4 x$" is required to be positive. It means,

$$\log_4 x > 0$$

Using the fact that if $\log_a x \geq y$, then $x \geq a^y$ for $a > 1$, we have :

$$\Rightarrow x > 4^0 \quad \Rightarrow x > 1$$

For "f(x)" to be real, "$\log_3 (\log_4 x)$" is required to be positive. It means,

$$\Rightarrow \log_3 (\log_4 x) > 0$$

$$\Rightarrow \log_4 x > 1$$

$$\Rightarrow x > 4^1 \quad \Rightarrow x > 4$$

Combining three intervals so obtained,

Domain of the function

Figure 7: Thick line represents domain of the given function.

$$\text{Domain} = (4, \infty)$$

Solution to Exercise (p. 7)

Hints : We need to find minimum and maximum value of logarithmic function for the values of x in domain of the function. The argument of logarithmic function is a quadratic function, whose coefficient of squared term is positive and D <0. It means its graph is a parabola opening up in the positive side of y-axis. The minimum value of the quadratic expression is :

$$y_{\text{min}} = -\frac{D}{4a}$$

$$y_{\text{max}} = \infty$$

Now, we know that graph of logarithmic function for base, a > 1, is a continuously increasing graph. It means that value of logarithmic function, corresponding to min and max values of quadratic expression is the range of given function.

$$\Rightarrow f\left(\frac{7}{4}\right) = \log 10\left(\frac{7}{14}\right)$$

$$\Rightarrow f\left(x \to \infty\right) \to \infty$$

Hence, range of given function is :

$$\text{Range} = \left(\log 10\left(\frac{7}{14}\right), \infty\right)$$

Solution to Exercise (p. 8)

Rearranging, we have :

$$\Rightarrow e^{f(x)} = e^x - e$$

Taking logarithm on either sides of equation,

$$\Rightarrow y = f(x) = \log\left(e^x - e\right)$$

For logarithmic function,

$$\Rightarrow e^x - e > 0 \quad \Rightarrow e^x > e \quad \Rightarrow x > 1$$

$$\text{Domain} = (1, \infty)$$

In order to find range, we solve function expression for y. In exponential form,

$$\Rightarrow e^y = e^x - e \quad \Rightarrow e^x = e^y - e$$

Taking logarithm on either sides of equation,

$$\Rightarrow x = \log_e\left(e^y - e\right)$$

For logarithmic function,

$$\Rightarrow e^y - e > 0 \quad \Rightarrow e^y > e \quad \Rightarrow y > 1$$

$$\text{Range} = (1, \infty)$$