

INCREASING AND DECREASING INTERVALS*

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A function is strictly increasing, strictly decreasing, non-decreasing and non-increasing in a suitably selected interval in the domain of the function. We have seen that a linear algebraic function maintains order of change throughout its domain. The order of change, however, may not be maintained for higher degree algebraic and other functions in its domain. We shall, therefore, determine monotonic nature in sub-intervals or domain as the case be.

One of the fundamental ways to determine nature of function is by comparing function values corresponding to two independent values (x_1 and x_2). This technique to determine nature of function works for linear and some simple function forms and is not useful for functions more complex in nature. In this module, we shall develop an algorithm based on derivative of function for determining nature of function in different intervals.

In the discussion about monotonic function in earlier module, we observed that order of change in function values is related to sign of the derivative of function. The task of finding increasing and decreasing intervals is, therefore, about finding sign of derivative of function in different intervals and determining points or intervals where derivative turns zero.

1 Nature of function and intervals

The steps for determining intervals are given as under :

- 1:** Determine derivative of given function i.e. $f'(x)$.
- 2:** Determine sign of derivative in different intervals.
- 3:** Determine monotonic nature of function in accordance with following categorization :

$f'(x) \geq 0$: equality holding for points only – strictly increasing interval

$f'(x) \geq 0$: equality holding for subsections also – non-decreasing or increasing interval

$f'(x) \leq 0$: equality holding for points only – strictly decreasing interval

$f'(x) \leq 0$: equality holding for subsections also – non-increasing or decreasing interval

5: The interval is open “()” at end points, if function is not continuous at end points. However, interval is close “[]” at end points, if function is continuous at end points.

In order to illustrate the steps, we consider a function,

$$f(x) = x^2 - x$$

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Its first derivative is :

$$f'(x) = 2x - 1$$

Here, critical point is $1/2$. First derivative, $f'(x)$, is positive for $x > 1/2$ and negative for $x < 1/2$. The signs of derivative are strict inequalities. It means that function is either strictly increasing or strictly decreasing in the open intervals. We know that infinity end is an open end. But, function is continuous in the given interval. Hence, we can include end point $x = 1/2$. Further, since derivative is zero at $x = 1/2$ i.e. at a single point, function remains strictly increasing or decreasing.

$$\text{Strictly increasing interval} = \left[-\infty, \frac{1}{2} \right]$$

$$\text{Strictly decreasing interval} = \left[\frac{1}{2}, \infty \right]$$

2 Algebraic functions

Derivative of algebraic function is also algebraic. In order to determine sign of derivative, we use sign scheme or wavy curve method, wherever expressions in derivative can be factorized.

Example 1

Problem : Determine monotonic nature of function in different intervals :

$$f(x) = 3x^4 - x^3$$

Solution : Its first derivative is :

$$\Rightarrow f'(x) = 12x^3 - 3x^2 = 3x^2(4x - 1)$$

Here, critical points are $0, 0, 1/4$. We have taken 0 twice as we need to write given function in terms of factors as :

$$\Rightarrow f'(x) = 3(x - 0)(x - 0)(4x - 1)$$

Since zero is repeated even times, derivative does not change at $x = 0$. The sign scheme is shown in the figure. First derivative, $f'(x)$, is positive for $x > 1/4$ and negative for $x < 1/4$. Derivative is zero at $x = 0$ and $1/4$ i.e. at points only. Clearly, the monotonic nature is "strict" in these intervals. But, function is continuous in the given interval. Hence, we include end point also :

Sign diagram

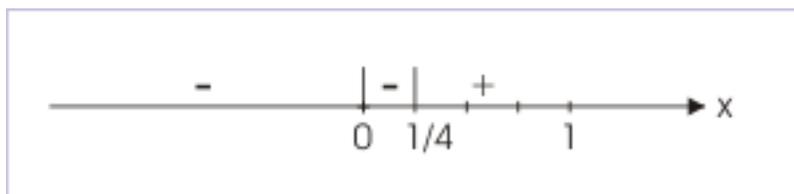


Figure 1: Increasing and decreasing intervals.

Strictly increasing interval = $\left[\frac{1}{4}, \infty\right)$

Strictly decreasing interval = $\left(-\infty, \frac{1}{4}\right]$

Example 2

Problem : Determine monotonic nature of function :

$$f(x) = \sqrt{\left(\frac{\pi^2}{4} - x^2\right)}$$

Solution : The domain of derivative, $f(x)$, is not \mathbf{R} . We first need to find its domain and then determine sign of its derivative within the domain. The expression in the square root in denominator is non-negative. Hence,

$$\Rightarrow \frac{\pi^2}{4} - x^2 \geq 0$$

$$\Rightarrow x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Its first derivative is :

$$\Rightarrow f'(x) = -\frac{2x}{2\sqrt{\left(\frac{\pi^2}{4} - x^2\right)}} = -\frac{x}{\sqrt{\left(\frac{\pi^2}{4} - x^2\right)}}$$

We know that square root is a positive number. It means that sign of derivative will solely depend on the sign of “x” in the numerator. Clearly, derivative is positive for $x < 0$ and negative for $x > 0$. Derivative is zero at $x = 0$ i.e. at a single point only. But, function is continuous in the given interval. Hence, we include end points as well :

Strictly increasing interval = $\left[-\frac{\pi}{2}, 0\right]$

Strictly decreasing interval = $\left[0, \frac{\pi}{2}\right]$

Example 3

Problem : Determine monotonic nature of function :

$$f(x) = 2x^3 + 3x^2 - 12x + 1$$

Solution : The first derivative of the given polynomial function is :

$$\Rightarrow f'(x) = 2 \times 3x^2 + 3 \times 2x - 12 = 6x^2 + 6x - 12$$

Clearly, the derivative is a quadratic function. We can determine the sign of the quadratic expression, using sign scheme for quadratic expression. Now, the roots of the corresponding quadratic equation when equated to zero is obtained as :

$$\Rightarrow 6x^2 + 6x - 12 = 0 \quad \Rightarrow x^2 + x - 2 = 0 \quad \Rightarrow x^2 + 2x - x - 2 = 0$$

$$\Rightarrow x(x+2) - 1(x+2) = 0 \quad \Rightarrow (x-1)(x+2) = 0$$

$$\Rightarrow x = 1, -2$$

Here, coefficient of “ x^2 ” is positive. Hence, sign of the middle interval is negative and side intervals are positive.

Intervals



Figure 2: Increasing and decreasing intervals

Since cubic polynomial is a continuous function, we can include end points also in the interval :

$$\text{Strictly increasing interval} = (-\infty, -2] \cup [1, \infty)$$

$$\text{Strictly decreasing interval} = [-2, 1]$$

3 Trigonometric function

Derivative of trigonometric function is also trigonometric function. We can determine nature of derivative in two ways. We use trigonometric values to determine nature of sign of first derivative. For this, we make use of sign and value diagram. Alternatively, we find zeroes of trigonometric function and knowing that some of these functions (sine, cosine etc.) changes sign across x-axis and are continuous functions, we can find sign of derivative as required.

Example 4

Problem : Determine sub-intervals of $[0, \pi/2]$ in which given function is (i) strictly increasing and (ii) strictly decreasing.

$$f(x) = \cos 3x$$

Solution : Its first derivative is :

$$f'(x) = -3\sin 3x$$

Corresponding to given interval $[0, \pi/2]$, argument to sine function is $[0, 3\pi/2]$. Sine function is positive in first two quadrants $[0, \pi]$ and negative in third quadrant $[\pi, 3\pi/2]$.

Corresponding to these argument values, $\sin 3x$ is positive in $[0, \pi/3]$ and negative in $[\pi/3, \pi/2]$ of the given interval. But, negative sign precedes $3\sin 3x$. Hence, derivative is negative in $[0, \pi/3]$ and positive in $[\pi/3, \pi/2]$ of the given interval.

Alternatively, we can find zero of $\sin 3x$ as :

$$-3\sin 3x = 0$$

$$\Rightarrow \sin 3x = 0$$

$$\Rightarrow 3x = n\pi; \quad n \in Z$$

$$x = \frac{n\pi}{3}; \quad n \in Z$$

Thus, there is one zero at $x = \pi/3$ for $n=1$, in the interval $(0, \pi/2)$. To test sign, we put $x = \pi/4$ in $-3\sin 3x$, we have $-3\sin 3\pi/4 < 0$. Hence, derivative is negative in $[0, \pi/3]$ and positive in $[\pi/3, \pi/2]$. Further since sine function is a continuous function, we include end points :

$$\text{Strictly decreasing interval} = \left[0, \frac{\pi}{3}\right]$$

$$\text{Strictly increasing interval} = \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$$

4 Logarithmic and exponential functions

Logarithmic and exponential functions are continuous in their domain intervals. Important point is that the derivative of exponential function is also an exponential function and it is always positive for base $a > 0$ and $a \neq 1$. This fact is also evident from its graphs. On the other hand, derivative of logarithmic function depends on the nature of its argument. The basic derivatives are :

$$f(x) = e^x \quad \Rightarrow \quad f'(x) = e^x$$

$$f(x) = \log_e x \quad \Rightarrow \quad f'(x) = \frac{1}{x}$$

Graphs of exponential functions

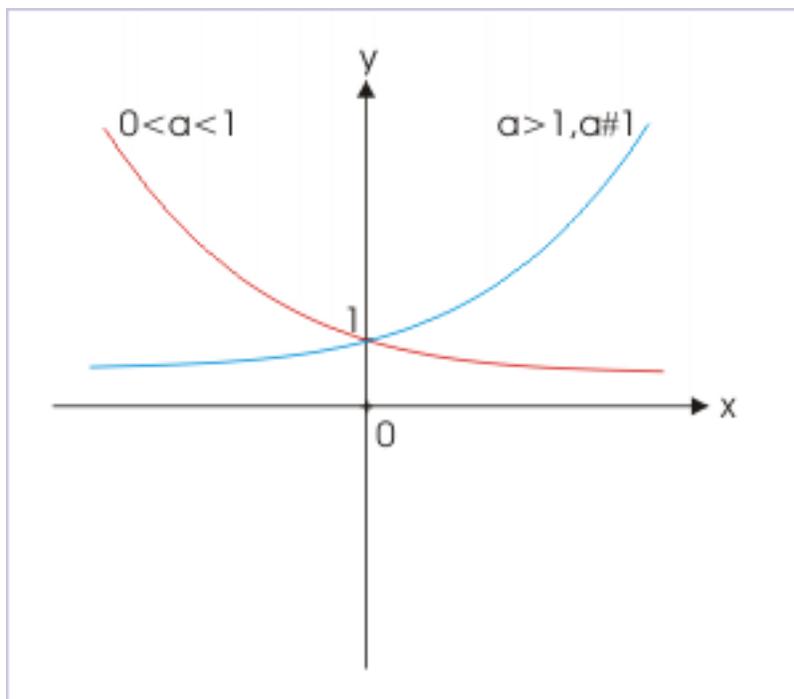


Figure 3: Graphs of exponential functions for different bases

Example 5

Problem : Determine monotonic nature of function :

$$f(x) = (1 + x)e^x$$

Solution : Its first derivative is :

$$f'(x) = e^x + (1 + x)e^x = (x + 2)e^x$$

Since e^x is positive for all values of x , the derivative is zero if :

$$(x + 2) = 0$$

$$\Rightarrow x = -2$$

For a test point, $x = -3$, $f'(x) = -1e^{-3} < 0$. Further, function is continuous in \mathbb{R} . Hence,

Strictly increasing interval = $[-2, \infty)$

Strictly decreasing interval = $(-\infty, -2]$

5 Exercises

Exercise 1

(Solution on p. 8.)

Determine monotonic nature of function :

$$f(x) = -2x^3 - 9x^2 + 12x + 7$$

Exercise 2

(Solution on p. 8.)

Determine sub-intervals of $[0, \pi/2]$ in which given function is (i) strictly increasing and (ii) strictly decreasing.

$$f(x) = \sin 3x$$

Exercise 3

(Solution on p. 8.)

Determine monotonic nature of function :

$$f(x) = (x - 1)e^x + 1$$

Solutions to Exercises in this Module

Solution to Exercise (p. 7)

Hint : Critical points are 1 and 2.

Strictly increasing interval = $[1, 2]$

Strictly decreasing interval = $(-\infty, 1] \cup [2, \infty)$

Solution to Exercise (p. 7)

Its first derivative is :

$$\Rightarrow f'(x) = 3\cos 3x$$

Corresponding to given interval $[0, \pi/2]$, argument to cosine function is $[0, 3\pi/2]$. Cosine function is positive in first quadrants $[0, \pi/2]$ and negative in second and third quadrant $[\pi/2, 3\pi/2]$. Corresponding to these argument values, $3\cos 3x$ is positive in $[0, \pi/6]$ and negative in $[\pi/6, \pi/2]$ of the given interval. Hence, derivative is positive in $[0, \pi/6]$ and negative in $[\pi/6, \pi/2]$.

Alternatively, we can find zero of $3\cos 3x$ as :

$$\cos 3x = 0$$

$$\Rightarrow 3x = (2n + 1) \frac{\pi}{2}; \quad n \in Z$$

$$\Rightarrow x = (2n + 1) \frac{\pi}{6}; \quad n \in Z$$

Thus, there is one zero, $x = \pi/6$, in the given interval $[0, \pi/2]$. To test sign, we put $x = \pi/4$ in $3\cos 3x$, we have $3\cos 3\pi/4 < 0$. Hence, derivative is positive in $[0, \pi/6]$ and negative in $[\pi/6, \pi/2]$.

Strictly increasing interval = $\left[0, \frac{\pi}{6}\right]$

Strictly decreasing interval = $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$

Solution to Exercise (p. 7)

Its first derivative is :

$$\Rightarrow f'(x) = e^x + (x - 1)e^x = e^x$$

Since e^x is positive for all values of x , the derivative is zero for all x . Hence, given function is strictly increasing in the interval of real number R .