Inverse functions (exercise)*

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Working rules
A. Onto function

- A function is onto if every image in the co-domain has a pre-image in the domain set.
- Range = Co-domain

B. Bijection

- If a function is both one-one and onto, then the function is a bijection.

C. Inverse function

- Solve given function for "x".
- Substitute "x" by inverse symbol "$f^{-1}(x)$" and "y" by "x" to find the rule of inverse function.
- Exchange the domain and range of the given function with that of inverse function.

1

Problem 1: A function $f : R \to R$ is given by :

$$f(x) = \sin(3x + 2)$$

Is the function surjective?
Solution :
Statement of the problem : Surjection is another name for onto function. Here, given function is a trigonometric function. We need to check whether its range is equal to co-domain.
According to question :

$$\text{Domain} = R$$

$$\text{Co-domain} = R$$

Now, let

$$y = f(x) = \sin(3x + 2)$$

Solving for "x", we have :

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$$\Rightarrow 5x + 2 = \sin^{-1} y$$

$$\Rightarrow x = \frac{\sin^{-1} y - 2}{5}$$

The inverse trigonometric function is valid in its domain of [-1,1]. Hence, values of "y", for which "x" is real, is [-1,1]. Therefore,

$$\text{Range} = [-1, 1]$$

Clearly, range is not equal to co-domain. Thus, given function is not an onto function.

2

Problem 2: A function $f : R \to R$ is given by :

$$f(x) = \frac{ax^2 + 6a - 8}{a + 6x - 8x^2}$$

Find the interval of values of "a" for which function is onto.

Solution :

Statement of the problem : It is given that the rational function is valid for all values of "x" as its domain is "R". It is also given that co-domain of the function is "R". In order to find the interval of "a", we consider that the function is an onto function. This implies that range of the function is equal to co-domain of the function i.e. "R".

Let

$$y = \frac{ax^2 + 6a - 8}{a + 6x - 8x^2}$$

$$\Rightarrow ax^2 + 6x - 8 - ay - 6xy + 8x^2 y = 0$$

Rearranging as quadratic equation in variable "x", we have :

$$\Rightarrow (a + 8y)\, x^2 + 6\,(1 - y)\, x - (8 + ay) = 0$$

Since "x" is real, discreminant of the quadratic equation should be non-negative,

$$\Rightarrow 36\,(1 - y)^{\,2} - 4X\,(a + 8y)\, X - (8 + ay) \geq 0$$

$$\Rightarrow 9(1 - y)^2 + (a + 8y)\, X\,(8 + ay) \geq 0$$

Expanding and rearranging as quadratic equation in variable "y", we have :

$$\Rightarrow 9\left(1 + y^2 - 2y\right) + \left(8a + a^2 y + 64y + 8ay^2\right) \geq 0$$

$$\Rightarrow (9 + 8a)\, y^2 + \left(a^2 + 46\right) y + (9 + 8a) \geq 0$$

But, we have assumed that function is an onto function. As such, range of the function is equal to co-domain i.e "R". It means that above inequality holds for all real values of "y". Now, remember (as explained for sign scheme of quadratic equation) that a quadratic expression evaluates to non-negative number if (i) coefficient of "y2" is a positive number and (ii) descreminant is a non-positive number. Hence,

$$\Rightarrow 9+8a>0 \quad \Rightarrow a>\ -9/8$$

and

$$\Rightarrow \left(a^2+46\right)^2-\{2\left(9+8a\right)\}^2\leq 0$$

$$\Rightarrow \left(a^2+46+18+16a\right)\left(a^2+46-18-16a\right)\leq 0$$

$$\Rightarrow \left(a^2+16a+64\right)\left(a^2-16a+28\right)\leq 0$$

$$\Rightarrow (a+8)^2\left(a-2a-14a+28\right)\leq 0$$

$$(a+8)^2(a-2)(a-14)\leq 0$$

$$\Rightarrow (a-2)(a-14)\leq 0$$

$$\Rightarrow 2\leq a\leq 14$$

The first condition earlier yielded as a > = 9/8. The required interval, therefore, is intersection of two intervals, which is [2,14].

3

Problem 3: Let "f" be a one-one function with domain [x,y,z] and range [1,2,3]. It is given that only one of three conditions given below are true and remaining two are false :

$$f(x)=1; \quad f(y)\neq 1; \quad f(z)\neq 2$$

Determine $f^{-1}(1)$.

Solution :

Statement of the problem : With three conditions, there are three different possibilities for one being true and other two being false. We shall check each such possibility to see whether function is onto function. If the function is onto, then the function is a bijection as it is already given that function is an injection.

First combination :

$$\text{If} \quad f(x)=1 \quad \text{is true, then} \quad f(x)=1.$$

$$\text{If} \quad f(y)\neq 1 \quad \text{is false, then} \quad f(y)=1.$$

$$\text{If} \quad f(z)\neq 2 \quad \text{is false, then} \quad f(z)=2.$$

The function is clearly not one-one and hence not a bijection as f(x) = f(y).

Second combination :

$$\text{If} \quad f(x)=1 \quad \text{is false, then} \quad f(x)=2 \quad \text{or} \quad 3.$$

$$\text{If} \quad f(y)\neq 1 \quad \text{is true, then} \quad f(y)=2 \quad \text{or} \quad 3.$$

$$\text{If} \quad f(z) \neq 2 \quad \text{is false, then} \quad f(z) = 2.$$

As f(z) = 2, f(x) = f(y) = 3. Again, the function is not one-one and hence not a bijection.
Third combination :

$$\text{If} \quad f(x) = 1 \quad \text{is false, then} \quad f(x) = 2 \quad \text{or} \quad 3.$$

$$\text{If} \quad f(y) \neq 1 \quad \text{is false, then} \quad f(y) = 1.$$

$$\text{If} \quad f(z) \neq 2 \quad \text{is true, then} \quad f(z) = 1 \quad \text{or} \quad 3.$$

As f(y) = 1, f(z) = 3 and f(x) = 2. In this case, we see that image and pre-image are related distinctly. The function is one-one and onto. Hence function is bijection for this combination. Also,

$$f^{-1}(1) = y$$

4

Problem 4: A function $f : R \to R$ is given by :

$$f(x) = x|x|$$

Find $f^{-1}(x)$.
Solution :
Statement of the problem : We first need to see that the given function is bijection. If the function is bijection, then we find the rule of corresponding inverse function following the algorithm given in the beginning of the module.
By definition of modulus function,

$$f(x) = xX - x = -x^2 \quad -\infty < x \leq 0$$

$$\Rightarrow f(x) = xXx = x^2 \quad 0 \leq x < \infty$$

We take the derivative of the function to check whether the function is an injection :

$$\Rightarrow f\prime(x) = -2x \quad -\infty < x \leq 0$$

$$\Rightarrow f\prime(x) = 2x \quad 0 \leq x < \infty$$

We see that derivative of function is non-negative number for all values of "x"

$$\Rightarrow f\prime(x) \geq 0$$

The equality holds only at a single point x = 0. Therefore, we conclude that function is an increasing function for all real values of "x". It means that given function is one-one function in the domain of "R". Now, we need to check whether function is onto function or not? For this, we find the range of the function. If range is "R", then range is equal to co-domain, which is also "R". For finding range of the function, we interpret the function, when "x" tends to become negative infinity or positive infinity (this is equivalent of taking limit).

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} -x^2 = -\infty$$

and

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x^2 = \infty$$

The range of the function, therefore, is set of real numbers, "R". Thus, we conclude that given function is an onto function.

The given function is both one-one and onto function and hence it is a bijection.

In order to find inverse function, we first solve the equation for "x" as :

For the interval, $-\infty < x \leq 0$

$$\Rightarrow y = -x^2 \quad \Rightarrow x^2 = -y \quad \Rightarrow x = \pm\sqrt{(-y)}$$

But, we see that value of "x" is non-positive in the specified interval. Hence,

$$\Rightarrow x = -\sqrt{(-y)}$$

For the interval, $\Rightarrow 0 \leq x < \infty$

$$\Rightarrow y = x^2 \quad \Rightarrow x^2 = y \quad \Rightarrow x = \pm\sqrt{(y)}$$

But, we see that value of "x" is non-negative in the specified interval. Hence,

$$\Rightarrow x = \sqrt{(y)}$$

Substituting "x" by "$f^{-1}(x)$" and "y" by "x", we have the rule of inverse function as :

$$f^{-1}(x) = -\sqrt{(-x)} \quad -\infty < x \leq 0$$

and

$$f^{-1}(x) = \sqrt{(x)} \quad 0 \leq x < \infty$$

5

Problem 5: A quadratic function is given as :

$$f(x) = x^2 + x + 1$$

Is the function invertible? If not, then find the intervals in which it is invertible. Also find corresponding inverse functions.

Solution :

Statement of the problem : The given function is a continuous function valid for all values of "x". We need to analyze function to determine whether function is bijection.

Let "x_1" and "x_2" be two values. Then,

$$x_1^2 + x_1 + 1 = x_2^2 + x_2 + 1 \quad \Rightarrow x_1^2 = x_2^2 \quad \Rightarrow x_1 = \pm x_2$$

It means function is not one-one, but many one function. For determining whether function is onto function, we investigate the nature of its derivative,

$$f\prime(x) = 2x + 1$$

Its root, when equated to zero, is -1/2. Its sign scheme is shown in the figure. The function value at "x = -1/2" corresponds to least value of the function.

Sign scheme of derivative

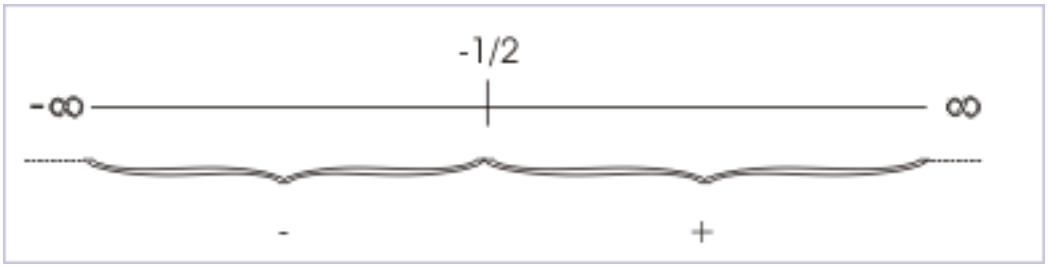

Figure 1: The function is strictly decreasing and increasing in two separate intervals.

$$\Rightarrow f\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) + 1 = \frac{1}{4} - \frac{1}{2} + 1 = \frac{3}{4}$$

From the figure it is clear that function is strictly decreasing in the interval (-∞,-1/2] and strictly increasing in the interval [-1/2, ∞). Further, we also observe from the graph as shown in the figure below that function is one-one in the individual interval.

Sign scheme of derivative

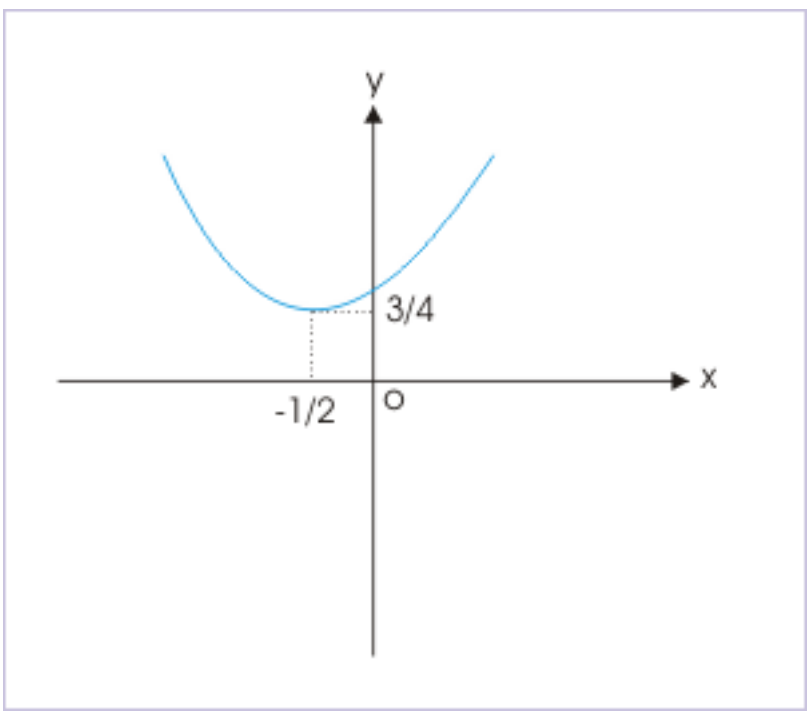

Figure 2: The function is strictly decreasing and increasing in two separate intervals.

Now, for determining inverse function, we solve the function for "x". Here :

$$y = f(x) = x^2 + x + 1$$

$$\Rightarrow x^2 + x + 1 - y = 0$$

Solving for "x", we have :

$$\Rightarrow x = \frac{-1 \pm \sqrt{(4y-3)}}{2}$$

Substituting "x" by "$\Rightarrow f^{-1}(x)$" and "y" by "x", we have the rule of inverse function as :
For interval $\Rightarrow \left(-\infty, -\frac{1}{2}\right]$

$$\Rightarrow f^{-1}(x) = \frac{-1 - \sqrt{(4x-3)}}{2}$$

For interval $\left[-\frac{1}{2}, \infty\right)$

$$\Rightarrow f^{-1}(x) = \frac{-1 + \sqrt{(4x-3)}}{2}$$

6

Problem 6: An onto function is given as :

$$f(x) = \log_a\{x + \sqrt{(x^2+1)}\}, a > 0, a \neq 1$$

Is the function invertible? If invertible, then find $f^{-1}(x)$.
Solution :
Statement of the problem : The given function is an onto function. This will be bijection i.e. invertible, if function is strictly monotonic (increasing or decreasing) so that function is one-one as well.

We see that $\sqrt{(x^2+1)} \geq 0$ for all values of "x". Thus, argument of the logarithmic function is positive for all real values of "x". It means that domain of the given function is "R". In order to determine monotonic nature of the function, we differentiate given function in relevant intervals. Before differentiating, we need to convert the base from "a" to "e" as :

$$f(x) = \log e\{x + \sqrt{(x^2+1)}\} X \log_a e$$

Differentiating with respect to independent variable,

$$\Rightarrow f\prime(x) = \frac{\log_a e}{\{x + \sqrt{(x^2+1)}\}} X \left[1 + \frac{2x}{\{x + \sqrt{(x^2+1)}\}}\right]$$

$$\Rightarrow f\prime(x) = \frac{\log_a e}{2\sqrt{(x^2+1)}}$$

Again $\sqrt{(x^2+1)} \geq 1$ i.e. a positive number. Hence, sign of f'(x) is same as that of

$$\log_a e$$

.

For $0 < a < 1$
See the graph. Here x = e = 2.718281828.

Sign scheme of derivative

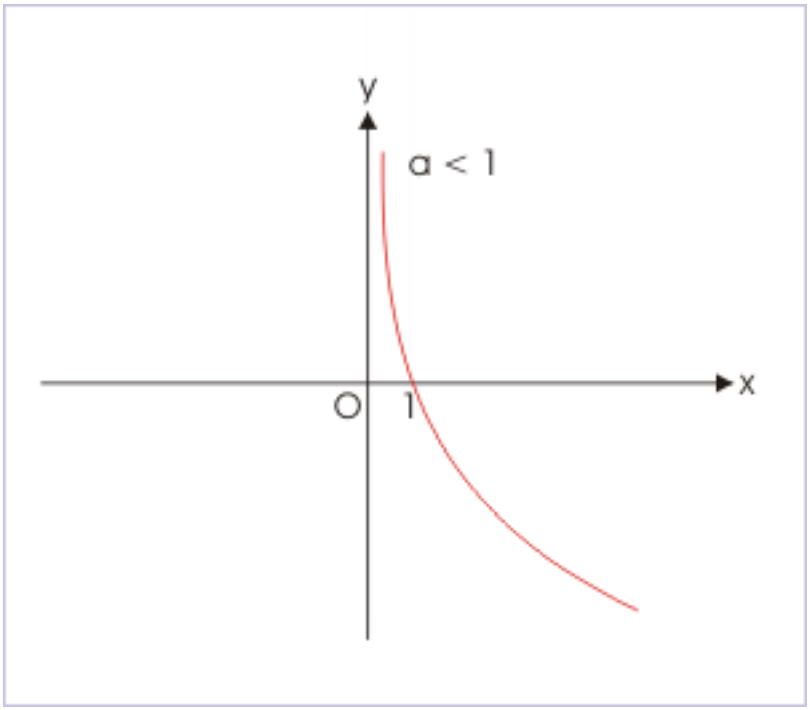

Figure 3: The function is strictly decreasing and increasing in two separate intervals.

$$\Rightarrow \log_a e < 0 \quad \Rightarrow f\prime(x) < 0$$

The function is a decreasing function in the domain i.e. "R"
For $a > 1$
See the graph. Here x = e = 2.718281828.

Plot of the function

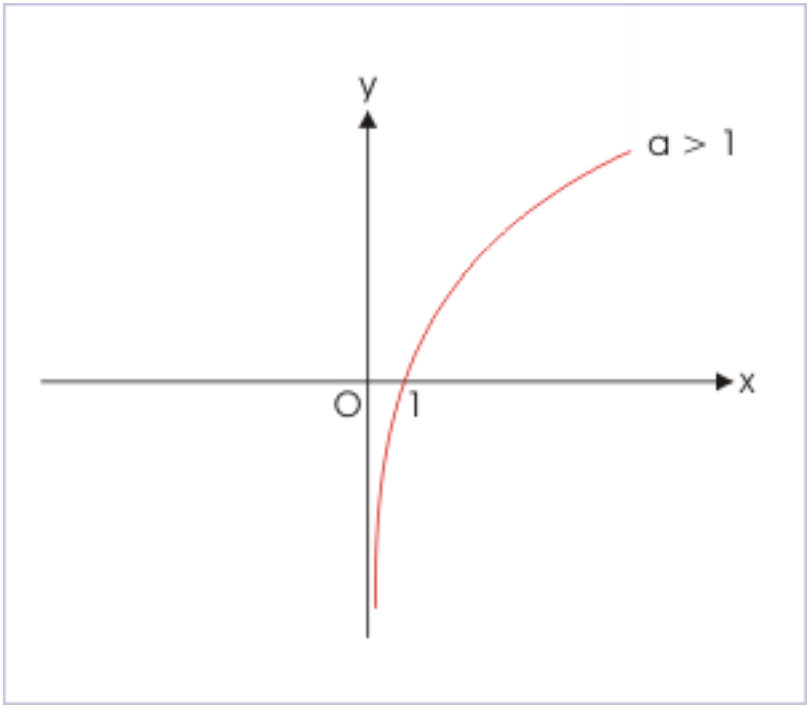

Figure 4: The function is strictly decreasing and increasing in two separate intervals.

$$\Rightarrow \log_a e > 0 \quad \Rightarrow f\prime(x) > 0$$

The function is an increasing function in the domain i.e. "R"

We see that function is either increasing or decreasing for all real values of "x". Thus, we conclude that function is strictly monotonic and function is on-one. This, in turn, means that function is bijection, which signifies that inverse of the function exists.

In order to find the inverse function, we write the logarithmic function in the equivalent exponential function as :

$$y = f(x) = \log_a\{x + \sqrt{(x^2+1)}\}$$

$$\Rightarrow a^y = x + \sqrt{(x^2+1)}$$

Also,

$$\Rightarrow a^{-y} = \frac{1}{a^y} = \frac{1}{x + \sqrt{(x^2+1)}}$$

$$\Rightarrow a^{-y} = \frac{\sqrt{(x^2+1)} - x}{\{x + \sqrt{(x^2+1)}\}\{\sqrt{(x^2+1)} - x\}}$$

$$\Rightarrow a^{-y} = \sqrt{(x^2+1)} - x$$

In order to solve for "x", we subtract two equations :

$$\Rightarrow a^y - a^{-y} = x + \sqrt{(x^2+1)} - \sqrt{(x^2+1)} + x = 2x$$

$$\Rightarrow x = \frac{1}{2}\left(a^y - a^{-y}\right)$$

Substituting "x" by "$f^{-1}(x)$ " and "y" by "x", we have the rule of inverse function as :

$$\Rightarrow f^{-1}(x) = \frac{1}{2}\left(a^x - a^{-x}\right)$$

7

Problem 7: A function $f : [1, \infty) \to [1, \infty)$ is given by :

$$f(x) = 2^{x(x-1)}$$

Find $f^{-1}(x)$.

Solution :

Statement of the problem : In order to determine the nature of given function to be a bijection, we need to check whether the function is both one-one and onto?

To check for one-one function, we determine the derivative of the function as :

$$f(x) = 2^{x(x-1)}$$

Taking logarithm on both sides,

$$\log_e(f(x)) = x(x-1)\log_e 2$$

Taking derivative on either side of the equation, we have :

$$\frac{f\prime(x)}{f(x)} = (2x-1)\log_e 2 \quad \Rightarrow f\prime(x) = f(x)(2x-1)\log_e 2$$

Now, "x" lie in the interval [1,∞). Hence, f(x) is a positive number. Also, $\log_e 2$ is a positive number. Therefore,

$$f\prime(x) = \text{positive number} X (2x-1)$$

The root of the expression of f'(x), when equated to zero, is "0.5". The derivative is positive in the interval between "1/2" and infinity. However, the domain of function begins at x = 1. We, therefore, conclude that the function is increasing in the interval given by [1,∞). The least value of the function is given as :

$$\Rightarrow f(1) = 2^{x(x-1)} = 2^0 = 1$$

We interpret the function, when "x" tends to become positive infinity (this is equivalent of taking limit).

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 2^{x(x-1)} = 2^{\infty X(\infty-1)} = 2^{\infty X \infty} = 2^{\infty} = \infty$$

Thus, range of the given function is [1, ∞). Clearly,

Range = Co-domain = [1, ∞)

Therefore, function is onto and hence bijection. It means that function is invertible. Now, we solve the function for "x" for finding inverse function. Taking log on the base of "2" on either side of the equation,

$$\Rightarrow \log_2 y = \log_2 x(x-1) = x(x-1) = x^2 - x$$

$$\Rightarrow x^2 - x - \log_2 y = 0$$

$$\Rightarrow x = \frac{1 \pm \sqrt{(1 + 4X1X\log_2 y)}}{2} = \frac{1 \pm \sqrt{(1 + 4\log_2 y)}}{2}$$

But, we know that domain of the function is x >= 1. Also, $\sqrt{(1 + 4\log_2 y)} \geq 1$. It means that only positive sign in the expression is valid.

$$\Rightarrow x = \frac{1 + \sqrt{(1 + 4\log_2 y)}}{2}$$

Substituting "x" by "$f^{-1}(x)$" and "y" by "x", we have the rule of inverse function as :

$$\Rightarrow f^{-1}(x) = \frac{1 + \sqrt{(1 + 4\log_2 x)}}{2}$$