Elliptic-Function Filter Properties*

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1 Elliptic-Function Filter Properties

In this section, a design procedure is developed that uses a Chebyshev error criterion in both the passband and the stopband. This is the fourth possible combination of Chebyshev and Taylor's series approximations in the passband and stopband. The resulting filter is called an elliptic-function filter, because elliptic functions are normally used to calculate the pole and zero locations. It is also sometimes called a Cauer filter or a rational Chebyshev filter, and it has equal ripple approximation error in both pass and stopbands [6], [5], [4], [7].

The error criteria of the elliptic-function filter are particularly well suited to the way specifications for filters are often given. For that reason, use of the elliptic-function filter design usually gives the lowest order filter of the four classical filter design methods for a given set of specifications. Unfortunately, the design of this filter is the most complicated of the four. However, because of the efficiency of this class of filters, it is worthwhile gaining some understanding of the mathematics behind the design procedure.

This section sketches an outline of the theory of elliptic- function filter design. The details and properties of the elliptic functions themselves should simply be accepted, and attention put on understanding the overall picture. A more complete development is available in [6], [3]. Straightforward design of elliptic-function filters can be accomplished by skipping this section and going directly to Program 8 in the appendix or by using Matlab. However, it is important to understand the basics of the underlying theory to use the packaged design programs intelligently.

Because both the passband and stopband approximations are over the entire bands, a transition band between the two must be defined. Using a normalized passband edge, the bands are defined by

$$0 < \omega < 1 \quad \text{passband} \tag{1}$$

$$1 < \omega < \omega_s \quad \text{transition band} \tag{2}$$

$$\omega_s < \omega < \infty \quad \text{stopband} \tag{3}$$

This is illustrated in Figure 1.

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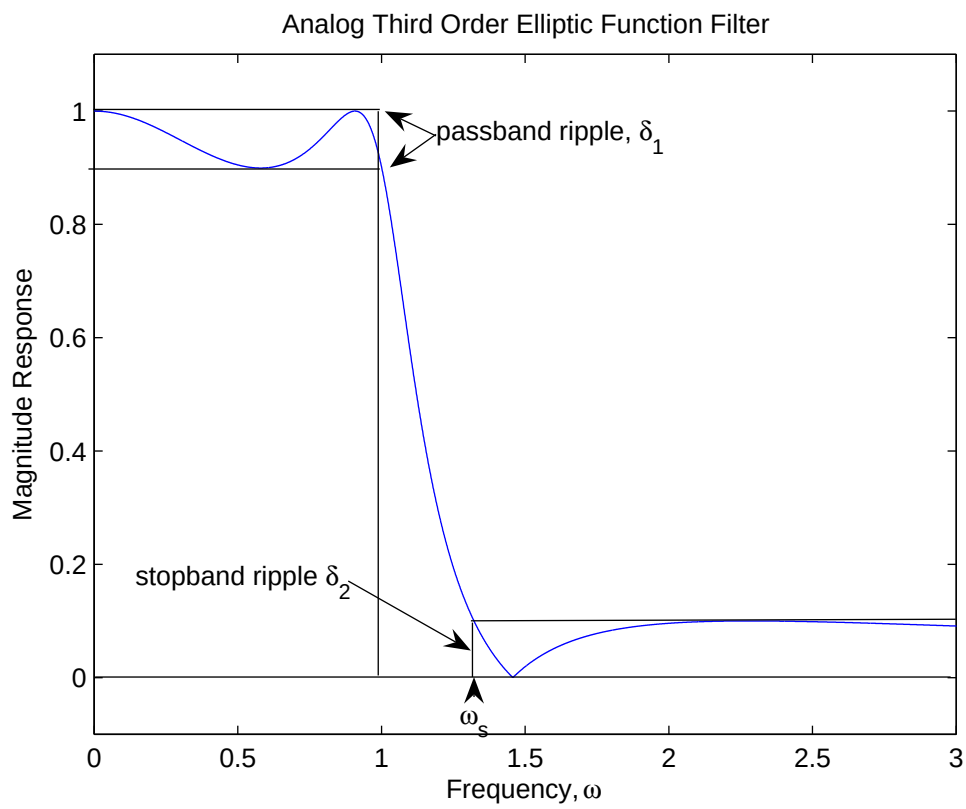

Figure 1: Third Order Analog Elliptic Function Lowpass Filter showing the Ripples and Band Edges

The characteristics of the elliptic function filter are best described in terms of the four parameters that specify the frequency response:

1. The maximum variation or ripple in the passband δ_1,
2. The width of the transition band $(\omega_s - 1)$,
3. The maximum response or ripple in the stopband δ_2, and
4. The order of the filter N.

The result of the design is that for any three of the parameters given, the fourth is minimum. This is a very flexible and powerful description of a filter frequency response.

The form of the frequency-response function is a generalization of that for the Chebyshev filter

$$FF(j\omega) = |F(j\omega)|^2 = \frac{1}{1 + \varepsilon^2 G^2(\omega)} \tag{4}$$

where

$$FF(s) = F(s)\,F(-s) \tag{5}$$

with $F(s)$ being the prototype analog filter transfer function similar to that for the Chebyshev filter. $G(\omega)$ is a rational function that approximates zero in the passband and infinity in the stopband. The definition of this function is a generalization of the definition of the Chebyshev polynomial.

1.1 Elliptic Functions

In order to develop analytical expressions for equal-ripple rational functions, an interesting class of transcendental functions, called the Jacobian elliptic functions, is outlined. These functions can be viewed as a generalization of the normal trigonometric and hyperbolic functions. The elliptic integral of the first kind [1] is defined as

$$u(\phi, k) = \int_0^{\phi} \frac{dy}{\sqrt{1 - k^2 sin^2(y)}} \tag{6}$$

The trigonometric sine of the inverse of this function is defined as the Jacobian elliptic sine of u with modulus k, and is denoted

$$sn(u, k) = sin(\phi(u, k)) \tag{7}$$

A special evaluation of (6) is known as the complete elliptic integral $K = u(\pi/2, k)$. It can be shown [1] that $sn(u)$ and most of the other elliptic functions are periodic with periods $4K$ if u is real. Because of this, K is also called the "quarter period". A plot of $sn(u, k)$ for several values of the modulus k is shown in Figure 2.

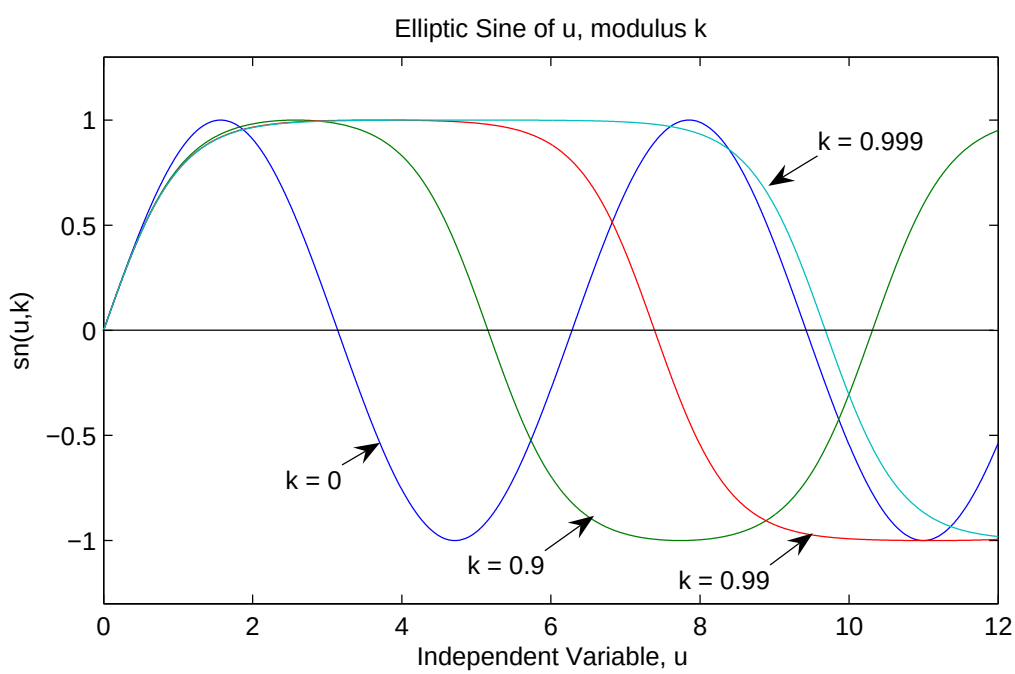

Figure 2: Jacobian Elliptic Sine Function of u with Modulus k

For k=0, $sn(u,0) = sin(u)$. As k approaches 1, the $sn(u,k)$ looks like a "fat" sine function. For $k = 1$, $sn(u,1) = tanh(u)$ and is not periodic (period becomes infinite).

The quarter period or complete elliptic integral K is a function of the modulus k and is illustrated in Figure 3.

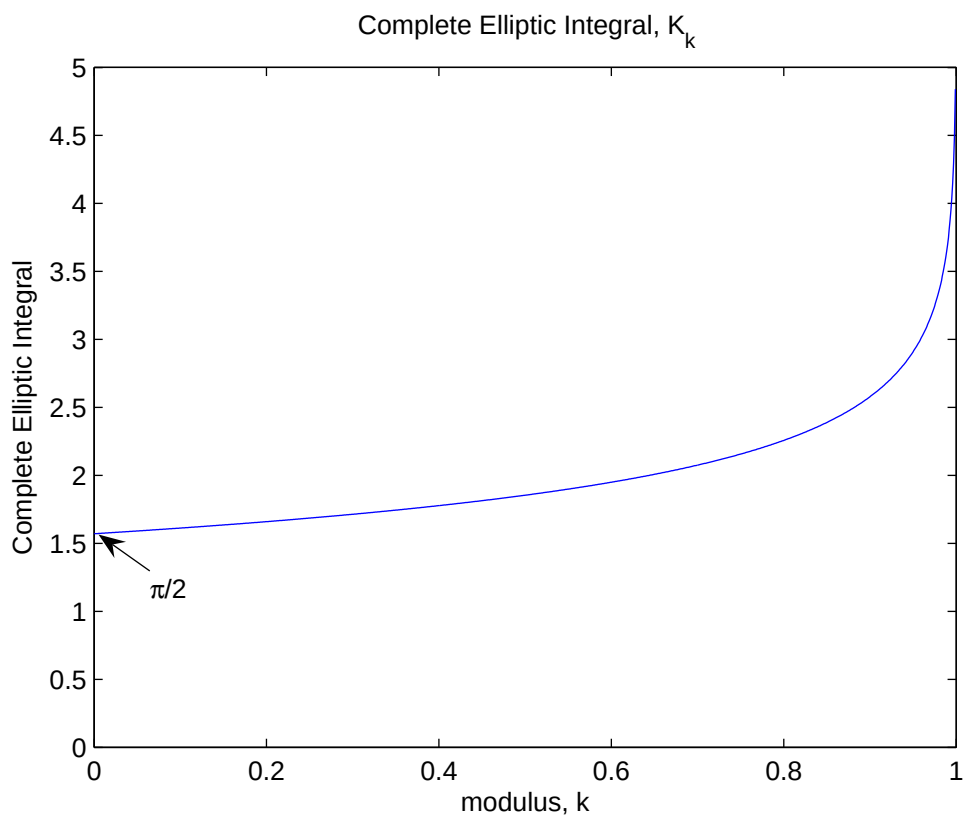

Figure 3: Complete Elliptic Integral as a function of the Modulus

For a modulus of zero, the quarter period is $K = \pi/2$ and it does not increase much until k nears unity. It then increases rapidly and goes to infinity as k goes to unity.

Another parameter that is used is the complementary modulus $k^{'}$ defined by

$$k^2 + k^{'2} = 1 \tag{8}$$

where both k and $k^{'}$ are assumed real and between 0 and 1. The complete elliptic integral of the complementary modulus is denoted $K^{'}$.

In addition to the elliptic sine, other elliptic functions that are rather obvious generalizations are

$$cn(u,k) = cos(\phi(u,k)) \tag{9}$$

$$sc(u,k) = tan(\phi(u,k)) \tag{10}$$

$$cs\left(u,k\right)=ctn\left(\phi\left(u,k\right)\right) \tag{11}$$

$$nc\left(u,k\right)=sec\left(\phi\left(u,k\right)\right) \tag{12}$$

$$ns\left(u,k\right)=csc\left(\phi\left(u,k\right)\right) \tag{13}$$

There are six other elliptic functions that have no trigonometric counterparts [1]. One that is needed is

$$dn\left(u,k\right)=\sqrt{1-k^2sn^2\left(u,k\right)} \tag{14}$$

Many interesting properties of the elliptic functions exist [1]. They obey a large set of identities such as

$$sn^2\left(u,k\right)+cn^2\left(u,k\right)=1 \tag{15}$$

They have derivatives that are elliptic functions. For example,

$$\frac{d\ sn}{du}=cn\ dn \tag{16}$$

The elliptic functions are the solutions of a set of nonlinear differential equations of the form

$$x^{''}+ax\pm bx^3=0 \tag{17}$$

Some of the most important properties for the elliptic functions are as functions of a complex variable. For a purely imaginary argument

$$sn\left(jv,k\right)=jsc\left(v,k^{'}\right) \tag{18}$$

$$cn\left(jv,k\right)=nc\left(v,k^{'}\right) \tag{19}$$

This indicates that the elliptic functions, in contrast to the circular and hyperbolic trigonometric functions, are periodic in both the real and the imaginary part of the argument with periods related to K and $K^{'}$, respectively. They are the only class of functions that are "doubly periodic".

One particular value that the sn function takes on that is important in creating a rational function is

$$sn\left(K+jK^{'},k\right)=1/k \tag{20}$$

1.2 The Chebyshev Rational Function

The rational function $G\left(\omega\right)$ needed in (4) is sometimes called a Chebyshev rational function because of its equal-ripple properties. It can be defined in terms of two elliptic functions with moduli k and k_1 by

$$G\left(\omega\right)=sn\ \left(n\ sn^{-1}\left(\omega,k\right),k_1\right) \tag{21}$$

In terms of the intermediate complex variable ϕ, $G\left(\omega\right)$ and ω become

$$G\left(\omega\right)=sn\left(n\phi,k_1\right) \tag{22}$$

$$\omega=sn\left(\phi,k\right) \tag{23}$$

It can be shown [3] that $G\left(\omega\right)$ is a real-valued rational function if the parameters k, k_1, and n take on special values. Note the similarity of the definition of $G\left(\omega\right)$ to the definition of the Chebyshev polynomial

$C_N(\omega)$. In this case, however, n is not necessarily an integer and is not the order of the filter. Requiring that $G(\omega)$ be a rational function requires an alignment of the imaginary periods [3] of the two elliptic functions in (22),(23). It also requires alignment of an integer multiple of the real periods. The integer multiplier is denoted by N and is the order of the resulting filter [3]. These two requirements are stated by the following very important relations:

$$nK^{'} = K_1^{'} \qquad \text{alignment of imaginary periods} \tag{24}$$

$$nK = NK_1 \qquad \text{alignment of a multiple of the real periods} \tag{25}$$

which, on removing the parameter n, become

$$\frac{K_1}{K}N = \frac{K_1^{'}}{K^{'}} \tag{26}$$

or

$$N = \frac{KK_1^{'}}{K^{'}K_1} \tag{27}$$

These relationships are central to the design of elliptic- function filters. N is an odd integer which is the order of the filter. For $N = 5$, the resulting rational function is shown in Figure 4.

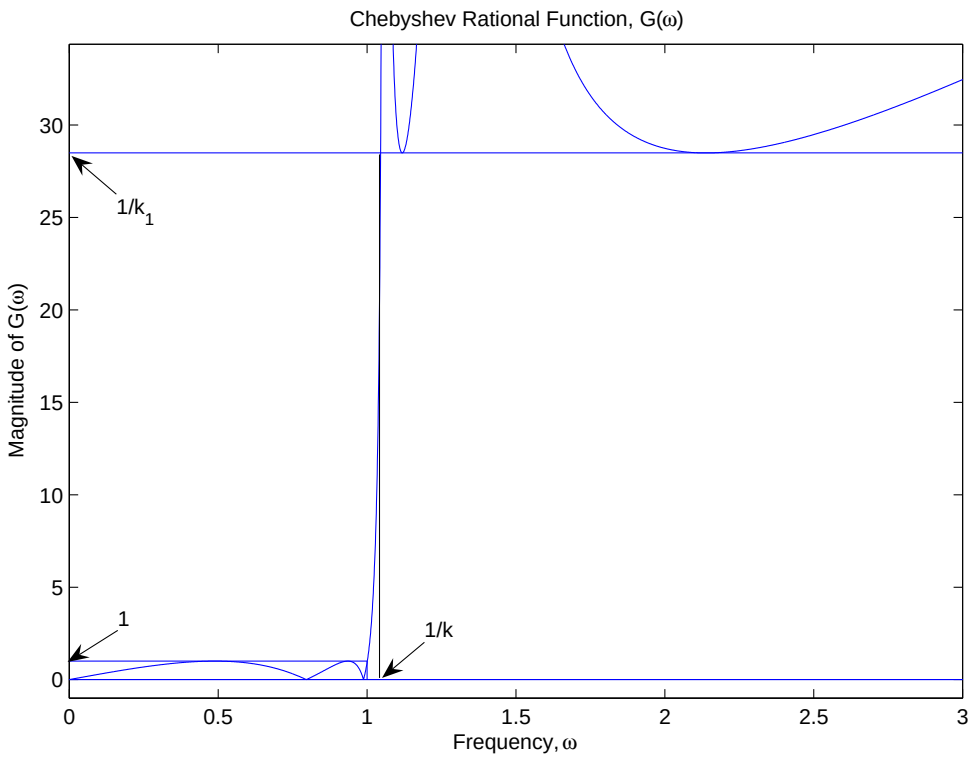

Figure 4: Fifth Order Elliptic Rational Function

This function is the basis of the approximation necessary for the optimal filter frequency response. It approximates zero over the frequency range $-1 < \omega < 1$ by an equal-ripple oscillation between ± 1. It also approximates infinity over the range $1/k < |\omega| < \infty$ by a reciprocal oscillation that keeps $|F(\omega)| > 1/k_1$. The zero approximation is normalized in both the frequency range and the $F(\omega)$ values to unity. The infinity approximation has its frequency range set by the choice of the modulus k, and the minimum value of $|F(\omega)|$ is set by the choice of the second modulus k_1.

If k and k_1 are determined from the filter specifications, they in turn determine the complementary moduli $k^{'}$ and $k_1^{'}$, which altogether determine the four values of the complete elliptic integral K needed to determine the order N in (27). In general, this sequence of events will not result in an integer. In practice, however, the next larger integer is used, and either k or k_1 (or perhaps both) is altered to satisfy (27).

In addition to the two-band equal-ripple characteristics, $G(\omega)$ has another interesting and valuable property. The pole and zero locations have a reciprocal relationship that can be expressed by

$$G(\omega)\, G(\omega_s/\omega) = 1/k_1 \tag{28}$$

where

$$\omega_s = 1/k \tag{29}$$

This states that if the zeros of $G(\omega)$ are located at ω_{zi}, the poles are located at

$$\omega_{pi} = 1/(k\omega_{zi}) \tag{30}$$

If the zeros are known, the poles are known, and vice versa. A similar relation exists between the points of zero derivatives in the 0 to 1 region and those in the $1/k$ to infinity region.

The zeros of $G(\omega)$ are found from (22) by requiring

$$G(\omega) = sn[n\phi, k_1] = 0 \tag{31}$$

which implies

$$n\phi = 2K_1 i \text{for}\, i = 0, 1, ... \tag{32}$$

From (21), this gives

$$\omega_{\mathrm{zi}} = \mathrm{sn}\left[2K_1 i/\mathrm{n}, \mathrm{k}\right], i = 0, 1, ... \tag{33}$$

This can be reformulated using (25) so that n and K_1 are not needed. For N odd, the zero locations are

$$\omega_{\mathrm{zi}} = \mathrm{sn}\left[2K_1 i/\mathrm{N}, \mathrm{k}\right], i = 0, 1, ... \tag{34}$$

The pole locations are found from these zero locations using (30). The locations of the zero-derivative points are given by

$$\omega_{di} = sn[K(2i+1)/N, k] \tag{35}$$

in the 0 to 1 region, and the corresponding points in the 1/k to infinity region are found from (30).

The above relations assume N to be an odd integer. A modification for N even is necessary. For proper alignment of the real periods, the original definition of $G(\omega)$ is changed to

$$G(\omega) = sn[\phi + K_1, k_1] \tag{36}$$

which gives for the zero locations with N even

$$\omega_{zi} = sn[(2i+1)K_1/n, k] \tag{37}$$

The even and odd N cases can be combined to give

$$\omega_{zi} = \pm sn(iK/N, k) \tag{38}$$

for

$$i = 0, 2, 4, ..., N-1 \quad \text{for N odd} \tag{39}$$

$$i = 1, 3, 5, ..., N-1 \quad \text{for N even} \tag{40}$$

with the poles determined from (30).

Note that it is possible to determine $G(\omega)$ from k and N without explicitly using k_1 or n. Values for k_1 and n are implied by the requirements of (29) or (28).

1.2.1 Zero Locations

The locations of the zeros of the filter transfer function $F(\omega)$ are easily found since they are the same as the poles of $G(\omega)$, given in (38).

$$\omega_{zi} = \frac{\pm 1}{k\ sn\left(iK/N, k\right)} \tag{41}$$

for

$$i = 0, 2, 4, ..., N-1 \quad \text{N odd} \tag{42}$$

$$i = 1, 3, 5, ..., N-1 \quad \text{N even} \tag{43}$$

These zeros are purely imaginary and lie on the ω axis.

1.2.2 Pole Locations

The pole locations are somewhat more complicated to find. An approach similar to that used for the Chebyshev filter is used here. $FF(s)$ becomes infinite when

$$1 + \varepsilon^2 G^2 = 0 \tag{44}$$

or

$$G = \pm j\left(1/\varepsilon\right) \tag{45}$$

Using (22) and the periodicity of sn (u,k) , this implies

$$G = sn\left(n\phi + 2K_1 i, k_1\right) = \pm j1/\varepsilon \tag{46}$$

or

$$\phi = \left(-2K_1 i + sn^{-1}\left(j1/e, k_1\right)\right)/n \tag{47}$$

Define ν_0 to be the second term in (47) by

$$j\nu_0 = \left(sn^{-1}\left(j1/e, k_1\right)\right)/n \tag{48}$$

which is similar to the equation for the Chebyshev case. Using properties of sn of an imaginary variable and (26), ν_0 becomes

$$\nu_0 = \left(K/NK_1\right) sc^{-1}\left(1/\varepsilon, k^{'}\right) \tag{49}$$

The poles are now found from (22),(23), (47), and (49) to be

$$s_{pi} = j\ sn\left(Ki/N + j\nu_0, k\right) \tag{50}$$

This equation can be more clearly written by using the summation formula [1] for the elliptic sine function to give

$$s_{pi} = \frac{cn\ dn\ sn^{'}\ cn^{'} + jsn\ dn^{'}}{1 - dn^2 sn^{'2}} \tag{51}$$

where

$$sn = sn\left(Ki/N, k\right), \quad cn = cn\left(Ki/N, k\right), \quad dn = dn\left(Ki/N, k\right) \tag{52}$$

$$sn^{'} = sn\left(\nu_0, k^{'}\right), \quad cn^{'} = cn\left(\nu_0, k^{'}\right), \quad dn^{'} = dn\left(\nu_0, k^{'}\right) \tag{53}$$

for

$$i = 0, 2, 4, ... \quad \text{N odd} \tag{54}$$

$$i = 1, 3, 5, ... \quad \text{N even} \tag{55}$$

The theory of Jacobian elliptic functions can be found in [1] and its application to filter design in [6], [3], [7]. The best techniques for calculating the elliptic functions seem to use the arithmetic-geometric mean; efficient algorithms are presented in [2]. A design program is given in [6] and a versitile FORTRAN program that is easily related to the theory in this chapter is given as Program 8 in the appendix of this book. Matlab has a powerful elliptic function filter design command as well as accurate algorithms for evaluating the Jacobian elliptic functions and integrals.

An alternative to the use of elliptic functions for finding the transfer function $F(s)$ pole locations is to obtain the zeros from (41), then find $G(\omega)$ using the reciprocal relation of the poles and zeros (30). $F(s)$ is constructed from $G(\omega)$ and ε from (4), and the poles are found using a root-finding algorithm. Another possibility is to find the zeros from (41) and the poles from the methods for finding a Chebyshev passband from arbitrary zeros. These approaches avoid calculating ν_0 by (49) or determining k from $K/K^{'}$, as is described in [2]. The efficient algorithms for evaluating the elliptic functions and the common use of powerful computers make these alternatives less attractive now.

1.2.3 Summary

In this section the basic properties of the Jacobian elliptic functions have been outlined and the necessary conditions given for an equal-ripple rational function to be defined in terms of them. This rational function was then used to construct a filter transfer function with equal-ripple properties. Formulas were derived to calculate the pole and zero locations for the filter transfer functions and to relate design specifications to the functions. These formulas require the evaluation of elliptic functions and are implemented in Program 8 in the appendix.

1.3 Elliptic-Function Filter Design Procedures

The equal-ripple rational function $G(\omega)$ is used to describe an optimal frequency-response function $F(j\omega)$ and to design the corresponding filter. The squared-magnitude frequency-response function is

$$|F(j\omega)|^2 = \frac{1}{1 + \varepsilon^2 G(\omega)^2} \tag{56}$$

with $G(\omega)$ defined by Jacobian Elliptic functions, and ε being a parameter that controls the passband ripple. The plot of this function for $N = 3$ illustrates the relation to the various specification parameters.

From Figure 1, it is seen that the passband ripple is measured by δ_1, the stopband ripple by δ_2, and the normalized transition band by ω_s. The previous section showed that

$$\omega_s = 1/k \tag{57}$$

which means that the width of the transition band determines k. It should be remembered that this development has assumed a passband edge normalized to unity. For the unnormalized case, the passband edge is ω_p and the stopband edge becomes

$$\omega_s = \frac{\omega_p}{k} \tag{58}$$

The stopband performance is described in terms of the ripple δ_2 normalized to a maximum passband response of unity, or in terms of the attenuation b in the stopband expressed in positive dB assuming a maximum passband response of zero dB. The stopband ripple and attenuation are determined from (56) and Figure 1 to be

$$\delta_2^2 = 10^{-b/10} = \frac{1}{1+\varepsilon^2/k_1^2} \tag{59}$$

This can be rearranged to give k_1 in terms of the stopband ripple or attenuation.

$$k_1^2 = \frac{\varepsilon^2}{1/\delta_2^2 - 1} = \frac{\varepsilon^2}{10^{b/10}-1} \tag{60}$$

The order N of the filter depends on k and k_1, as shown in (27). Equations (58), (60), and (27) determine the relation of the frequency-response specifications and the elliptic-function parameters. The location of the transfer function poles and zeros must then be determined.

Because of the required relationships of (27) and the fact that the order N must be an integer, the passband ripple, stopband ripple, and transition band cannot be independently set. Several straightforward procedures can be used that will always meet two of the specifications and exceed the third.

The first design step is generally the determination of the order N from the desired passband ripple δ_1, the stopband ripple δ_2, and the transition band controlled by ω_s. The following formulas determine the moduli k and k_1 from the passband ripple δ_1 or its dB equavilent a, and the stopband ripple δ_2 or its dB attenuation equivalent b:

$$\varepsilon = \sqrt{\frac{2\delta_1 - \delta_1^2}{1-2\delta_1-\delta_1^2}} = \sqrt{10^{a/10}-1} \tag{61}$$

$$k_1 = \frac{\varepsilon}{\sqrt{1/\delta_2^2 - 1}} = \frac{\varepsilon}{\sqrt{10^{b/10}-1}} \tag{62}$$

$$k_1^{'} = \sqrt{1-k_1^2} \tag{63}$$

$$k = \omega_p/\omega_s \qquad k^{'} = \sqrt{1-k^2} \tag{64}$$

The order N is the smallest integer satisfying

$$N \geq \frac{KK_1^{'}}{K^{'}K_1} \tag{65}$$

This integer order N will not in general exactly satisfy (27), i.e., will not satisfy (27) with equality. Either k or k_1 must to recalculated to satisfy (27) and (65). The various possibilities for this are developed below.

1.4 Methods for Meeting Specifications

1.4.1 Fixed Order, Passband Ripple, and Transition Band

Given N from (65) and the specifications δ_1, ω_p, and ω_s, the parameters ε and k are found from (62) and (refcc50). From k, the complete elliptic integrals K and K' are calculated [2]. From (27), the ratio $K/K^{'}$ determines the ratio $K_1^{'}/K_1$. Using numerical methods from [1], k_1 is calculated. This gives the desired δ_1, ω_p, and ω_s and minimizes the stopband ripple δ_2 (or maximizes the stopband attenuation b).

Using these parameters, the zeros are calculated from (refcc31) and the poles from (refcc39). Note the zero locations do not depend on ε or k_1, but only on N and ω_s. This makes the tradeoff between stop and passband occur in (refcc48) and only affects the calculation of nu_0 in (refcc38)

This approach which minimizes the stopband ripple is used in the IIR filter design program in the appendix of this book.

1.4.2 Fixed Order, Stopband Rejection, and Transition Band

Given N from (65) and the specifications δ_2, ω_p, and ω_s, the parameter k is found from (refcc50). From k, the complete elliptic integrals K and K' are calculated [2]. From (27), the ratio K/K' determines the ratio $K_1^{'}/K_1$. Using numerical methods from [1], k_1 is calculated. From k_1 and δ_2, ε and δ_1 are found from

$$\varepsilon = k_1\sqrt{1/\delta_2^2 - 1} \tag{66}$$

and

$$\delta_1 = 1 - \frac{1}{\sqrt{1+\varepsilon^2}} \tag{67}$$

This set of parameters gives the desired ω_p, ω_s, and stopband ripple and minimizes the passband ripple. The zero and pole locations are found as above.

1.4.3 Fixed Order, Stopband, and Passband Ripple

Given N from (65) and the specifications δ_1, δ_2, and either ω_p or ω_s, the parameters ε and k_1 are found from (62) and (refcc48). From k_1, the complete elliptic integrals K_1 and $K_1^{'}$ are calculated [2]. From (27), the ratio $K_1/K_1^{'}$ determines the ratio $K^{'}/K$. Using numerical methods from [1], k is calculated. This gives the desired passband and stopband ripple and minimizes the transition-band width. The pole and zero locations are found as above.

1.4.4 An Approximation

In many filter design programs, after the order N is found from (65), the design proceeds using the original e, k, and k_1, even though they do not satisfy (27). The resulting design has the desired transition band, but both pass and stopband ripple are smaller than specified. This avoids the calculation of the modulus k or k_1 from a ratio of complete elliptic integrals as was necessary in all three cases above, but produces results that are difficult to exactly predict.

Example 1: Design of a Third-Order Elliptic-Function Filter

A lowpass elliptic-function filter is desired with a maximum passband ripple of $\delta_1 = 0.1$ or $a = 0.91515$ dB, a maximum stopband ripple of $\delta_2 = 0.1$ or $b = 20$ dB rejection, and a normalized stopband edge of $\omega_s = 1.3$ radians per second. The first step is to determine the order of the filter.

From ω_s, the modulus k is calculated and then the complimentary modulus using the relations in (refcc50). Special numerical algorithms illustrated in Program 8 are then used to find the complete elliptic integrals K and $K^{'}$[2].

$$k = 1/1.3 = 0.769231, \qquad k^{'} = \sqrt{1-k^2} = 0.638971 \tag{68}$$

$$K = 1.940714, \qquad K^{'} = 1.783308 \tag{69}$$

From δ_1, ε is calculated using (62), and from ε and δ_2, k_1 is calculated from (refcc48). $k_1^{'}$, K_1, and $K_1^{'}$ are then calculated.

$$\varepsilon = 0.4843221 \qquad \text{as for the Chebyshev example.} \tag{70}$$

$$k_1 = 0.0486762, \qquad k_1^{'} = 0.9988146 \tag{71}$$

$$K_1 = 1.571727, \qquad K_1^{'} = 4.4108715 \tag{72}$$

The order is obtained from (27) by calculating

$$\frac{K\ K^{'}}{K^{'}\ K_1} = 3.0541 \tag{73}$$

This is close enough to 3 to set $N = 3$. Rather than recalculate k and k_1, the already calculated values are used as discussed in the design method D in this section. The zeros are found from (refcc31) using only N and k from above.

$$\omega_z = \frac{\pm 1}{k\ sn\left(2K/N, k\right)} = \pm 1.430207 \tag{74}$$

To find the pole locations requires the calculation of ν_0 from (refcc38) which is somewhat complicated. It is carried out using the algorithms in Program 8 in the appendix.

$$\nu_0 = \frac{K}{N\ K_1} sc^{-1}\left(1/\varepsilon, k_1^{'}\right) = 0.6059485 \tag{75}$$

From this value of ν_0, and k and N above, the elliptic functions in (refcc40) are calculated to give

$$sn^{'} = .557986, \quad cn^{'} = 0.829850, \quad dn^{'} = 0.934281 \tag{76}$$

which, for the single real pole corresponding to i = 0 in (refcc39), gives

$$s_p = 0.672393 \tag{77}$$

For the complex conjugate pair of poles corresponding to $i = 2$, the other elliptic functions in (refcc40) are

$$sn = 0.908959, \quad cn = 0.416886, \quad dn = 0.714927 \tag{78}$$

which gives from (refcc39) for the poles

$$s_p = 0.164126 \pm j1.009942 \tag{79}$$

The complete transfer function is

$$F\left(s\right) = \frac{s^2 + 2.045492}{\left(s + 0.672393\right)\left(s^2 + 0.328252s + 1.046920\right)} \tag{80}$$

This design should be compared to the Chebyshev and inverse- Chebyshev designs.

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