Central Limit Theorem: Central Limit Theorem for Sample Means*

Susan Dean
Barbara Illowsky, Ph.D.

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Suppose X is a random variable with a distribution that may be known or unknown (it can be any distribution). Using a subscript that matches the random variable, suppose:

a. μ_X = the mean of X
b. σ_X = the standard deviation of X

If you draw random samples of size n, then as n increases, the random variable $\overline{X}$ which consists of sample means, tends to be **normally distributed** and
$\overline{X} \sim N\left(\mu_X, \frac{\sigma_X}{\sqrt{n}}\right)$

The Central Limit Theorem for Sample Means says that if you keep drawing larger and larger samples (like rolling 1, 2, 5, and, finally, 10 dice) and **calculating their means** the sample means form their own **normal distribution** (the sampling distribution). The normal distribution has the same mean as the original distribution and a variance that equals the original variance divided by n, the sample size. n is the number of values that are averaged together not the number of times the experiment is done.

To put it more formally, if you draw random samples of size n,the distribution of the random variable $\overline{X}$, which consists of sample means, is called the **sampling distribution of the mean**. The sampling distribution of the mean approaches a normal distribution as n, the sample size, increases.

The random variable $\overline{X}$ has a different z-score associated with it than the random variable X. $\overline{x}$ is the value of $\overline{X}$ in one sample.

$$z = \frac{\overline{x} - \mu_X}{\left(\frac{\sigma_X}{\sqrt{n}}\right)} \tag{1}$$

μ_X is both the average of X and of $\overline{X}$.
$\sigma_{\overline{X}} = \frac{\sigma_X}{\sqrt{n}}$ = standard deviation of $\overline{X}$ and is called the **standard error of the mean.**

Example 1
An unknown distribution has a mean of 90 and a standard deviation of 15. Samples of size $n = 25$ are drawn randomly from the population.

Problem 1
Find the probability that the **sample mean** is between 85 and 92.

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Solution

Let X = one value from the original unknown population. The probability question asks you to find a probability for the **sample mean**.

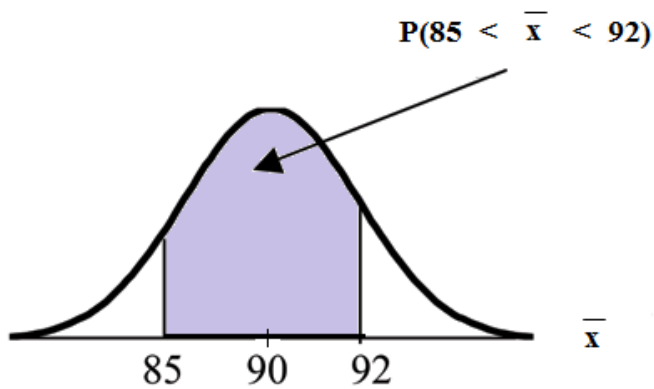
Let $\overline{X}$ = the mean of a sample of size 25. Since $\mu_X = 90$, $\sigma_X = 15$, and $n = 25$;

then $\overline{X} \sim N\left(90, \frac{15}{\sqrt{25}}\right)$

Find $P\left(85 < \overline{x} < 92\right)$ Draw a graph.

$P\left(85 < \overline{x} < 92\right) = 0.6997$

The probability that the sample mean is between 85 and 92 is 0.6997.



TI-83 or 84: `normalcdf`(lower value, upper value, mean, standard error of the mean)

The parameter list is abbreviated (lower value, upper value, μ, $\frac{\sigma}{\sqrt{n}}$)

`normalcdf` $\left(85, 92, 90, \frac{15}{\sqrt{25}}\right) = 0.6997$

Problem 2

Find the value that is 2 standard deviations above the expected value (it is 90) of the sample mean.

Solution

To find the value that is 2 standard deviations above the expected value 90, use the formula

value $= \mu_X + (\#\text{ofSTDEVs})\left(\frac{\sigma_X}{\sqrt{n}}\right)$

value $= 90 + 2 \cdot \frac{15}{\sqrt{25}} = 96$

So, the value that is 2 standard deviations above the expected value is 96.

Example 2

The length of time, in hours, it takes an "over 40" group of people to play one soccer match is normally distributed with a **mean of 2 hours** and a **standard deviation of 0.5 hours**. A **sample of size** n = **50** is drawn randomly from the population.

Problem

Find the probability that the **sample mean** is between 1.8 hours and 2.3 hours.

Solution

Let X = the time, in hours, it takes to play one soccer match.

The probability question asks you to find a probability for the **sample mean time, in hours**, it takes to play one soccer match.

Let $\overline{X}$ = the **mean** time, in hours, it takes to play one soccer match.

If μ_X = _____, σ_X = _____, and n = _____, then $\overline{X} \sim N($_____, _____$)$ by the Central Limit Theorem for Means.

μ_X =**2**, σ_X =**0.5**, n =**50**, and $X \sim N\left(2\ ,\ \frac{0.5}{\sqrt{50}}\right)$

Find $P\left(1.8 < \overline{x} < 2.3\right)$. Draw a graph.

$P\left(1.8 < \overline{x} < 2.3\right) = 0.9977$

`normalcdf` $\left(1.8, 2.3, 2, \frac{.5}{\sqrt{50}}\right)\ =\ 0.9977$

The probability that the mean time is between 1.8 hours and 2.3 hours is _____.

Glossary

Definition 1: Average

A number that describes the central tendency of the data. There are a number of specialized averages, including the arithmetic mean, weighted mean, median, mode, and geometric mean.

Definition 2: Central Limit Theorem

Given a random variable (RV) with known mean μ and known standard deviation σ. We are sampling with size n and we are interested in two new RVs - the sample mean, $\overline{X}$, and the sample sum, ΣX. If the size n of the sample is sufficiently large, then $\overline{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$ and $\Sigma X \sim N\left(\mathrm{n}\mu, \sqrt{n}\sigma\right)$. If the size n of the sample is sufficiently large, then the distribution of the sample means and the distribution of the sample sums will approximate a normal distribution regardless of the shape of the population. The mean of the sample means will equal the population mean and the mean of the sample sums will equal n times the population mean. The standard deviation of the distribution of the sample means, $\frac{\sigma}{\sqrt{n}}$, is called the standard error of the mean.

Definition 3: Normal Distribution

A continuous random variable (RV) with pdf f(x) = $\frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)2}/2\sigma^2$, where μ is the mean of the distribution and σ is the standard deviation. Notation: $X \sim N\left(\mu, \sigma\right)$. If $\mu = 0$ and $\sigma = 1$, the RV is called **the standard normal distribution**.

Definition 4: Standard Error of the Mean

The standard deviation of the distribution of the sample means, $\frac{\sigma}{\sqrt{n}}$.