Fast wavelet transform*

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1 One-dimensional wavelet transform

Suppose we are given as signal the projection of a function onto the space V_{j+1}:

$$\mathcal{P}_{j+1}f = \sum_k s_{j+1,k}\varphi_{j+1,k}(x), \quad s_{j+1,k} =< f, \tilde{\varphi}_{j+1,k} > \ . \tag{1}$$

Using the dual refinement equations, we have:

$$\begin{aligned} s_{j,k} =< f, \tilde{\varphi}_{j,k} > &= < f, \textstyle\sum_l \tilde{h}_l \varphi_{j+1,2k+l} > \\ &= \textstyle\sum_k \tilde{h}_{l-2k} s_{j+1,l} \ , \end{aligned} \tag{2}$$

where the coefficients s_{jk} are called **scaling coefficients**, since they are related to scaling functions. Similarly, the **wavelet** or **detail coefficients**d_{jk} are obtained as

$$d_{jk} =< f, \tilde{\psi}_{jk} >= \sum_k \tilde{g}_{l-2k} s_{j+1,l} \ . \tag{3}$$

The coefficients s_{jk} and d_{jk} are obtained from $s_{j+1,l}$ by 'moving average' schemes, using the filter coefficients $\{\tilde{h}_l\}$ and $\{\tilde{g}_l\}$ as 'weights', with the exception that these moving averages are sampled only at the even integers, i.e. a downsampling is performed. Such transform allows, once we have computed $s_{J,k}=< f, \tilde{\varphi}_{J,k} >$ for a fine level $J \in \mathbb{N}$, to compute s_{jk} and d_{jk} for all coarser levels $j < J$ without evaluating the integrals.

Suppose now we are given the values of f at $n = 2^J$ equispaced design points. The scaling functions $\tilde{\varphi}_{J,k}, k = 0, ..., 2^J - 1$, are compactly supported and localized around $2^{-J}k$. Hence the coefficients $< f, \tilde{\varphi}_{J,k} >$ are weighted and scaled average of f on a neighborhood of $2^{-J}k$ which becomes smaller as J tends to infinity. Consequently, it makes sense to replace the integral $< f, \tilde{\varphi}_{J,k} >$ by the (scaled) value of f at $2^{-J}k$. More complicate quadrature formulae have been developed in [6], [7], [8].

With $s_j := \{s_{jk}; k = 0, ..., 2^j - 1\}$ and $d_j := \{d_{jk}; k = 0, ..., 2^j - 1\}$, the forward (or analyzing) wavelet transform given by (2)-(3) can be rewritten as

$$s_j = \tilde{H}_j^* s_{j+1} \quad \text{and} \quad d_j = \tilde{G}_j^* s_{j+1} \ , \tag{4}$$

where $\tilde{H}_j^*$ denotes the Hermitian conjugate of $\tilde{H}_j$.

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The inverse (or synthesis) transform is found by using the primal refinement equations and the fact that $V_{j+1} = V_j \oplus W_j$.

$$\begin{aligned} \mathcal{P}_{j+1} f &= \textstyle\sum_l s_{j+1,l}\varphi_{j+1,l} = \sum_k s_{j,k}\varphi_{j,k} + \sum_k d_{j,k}\psi_{j,k} \\ &= \textstyle\sum_k s_{j,k} \sum_l h_l \varphi_{j+1,2k+l} + \sum_k d_{j,k} \sum_l g_l \varphi_{j+1,2k+l} \\ &= \textstyle\sum_l \varphi_{j+1,l} \left(\sum_k h_{l-2k} s_{j,k} + \sum_k g_{l-2k} d_{jk}\right) \ , \end{aligned} \tag{5}$$

from which it follows that

$$s_{j+1,l} = \sum_k h_{l-2k} s_{jk} + \sum_k g_{l-2k} d_{jk} \ . \tag{6}$$

In matrix form, we have

$$s_{j+1} = H_j s_j + G_j d_j \ . \tag{7}$$

In the finite and classical setting, the matrices H_j, G_j, $\tilde{H}_j$ and $\tilde{G}_j$ are of size $2^{j+1} \times 2^j$. Moreover, if the basis functions are compactly supported, the four filters (h_l, g_l, $\tilde{h}_l$, $\tilde{g}_l$) have only a finite number of nonzero elements, and hence all these matrices are banded.

1.1 Example: Haar wavelet transform

In case of the orthogonal Haar transform, $\tilde{H}_j^* = H_j^*$ and is of the form

$$\tilde{H}_j^* = \begin{bmatrix} h_0 & h_1 & & & & \\ & & h_0 & h_1 & & \\ & & & & \dots & \\ & & & & h_0 & h_1 \end{bmatrix} \tag{8}$$

since only h_0 and h_1 are different from zero : $h_0 = h_1 = 1/\sqrt{2}$. The high-pass filter $\{g_l\}$ is such that $g_0 = -1/\sqrt{2}$ and $g_1 = 1/\sqrt{2}$. The forward transform (2)-(3) reduces to

$$\begin{aligned} s_{j,k} &= \tfrac{1}{\sqrt{2}} s_{j+1,2k+1} + \tfrac{1}{\sqrt{2}} s_{j+1,2k} \\ d_{j,k} &= \tfrac{1}{\sqrt{2}} s_{j+1,2k+1} - \tfrac{1}{\sqrt{2}} s_{j+1,2k} \ , \end{aligned} \tag{9}$$

and the reconstruction is given by

$$\begin{aligned} s_{j+1,2k} &= \tfrac{1}{\sqrt{2}} s_{j,k} - \tfrac{1}{\sqrt{2}} d_{j,k} \\ s_{j+1,2k+1} &= \tfrac{1}{\sqrt{2}} s_{j,k} + \tfrac{1}{\sqrt{2}} d_{j,k} \ . \end{aligned} \tag{10}$$

2 Two-dimensional wavelet transform

The wavelet transform has been successfully applied to compress images, which are modelled as functions defined on a regular two-dimensional grid.

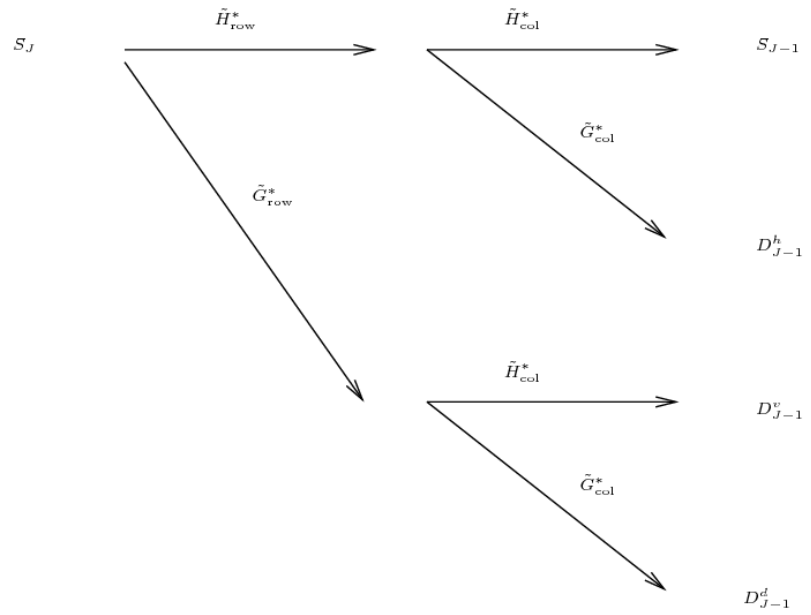

Figure 1: Two-dimensional wavelet transform: first the filters are applied on the column of the matrix S_J, this produces two matrices. The filters are applied a second time on the columns of these two matrices, resulting in four elements: a matrix of scaling coefficients, and three detail matrices.

The easiest way to build a two-dimensional MRA is probably to use tensor products of spaces, see [2], [4]. In terms of wavelet transforms, this leads to applying two times a one-dimensional transform: first on the 'row' of the signal matrix S_J, and second on the 'columns' of the resulting two matrices, see Figure 1. In this figure, we see that, at each level of the decomposition, three types of detail coefficients are produced: D^h_j, D^v_j and D^d_j. These superscripts recall that, in an image, **horizontal** edges will lead to large values of D^h_j, **vertical** edges will show up in D^v_j and D^d_j will be sensitive to **diagonal** lines.

However, such a transform is not able to compress efficiently an image that contains curves. More complex bidimensional bases are now proposed in the literature to better model discontinuities along curves, see for example [1], [3], [5], [9].

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