# FUNCTION CONCEPTS – ALGEBRAIC GENERALIZATIONS<sup>\*</sup>

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#### Abstract

This module discusses the generalizations of algebraic functions and their implications.

When you have a "generalization," you have one **broad fact** that allows you to assume many **specific** facts as examples.

### Example 1

Generalization: "Things fall down when you drop them."

### Specific facts, or examples:

- Leaves fall down when you drop them
- Bricks fall down when you drop them
- Tennis balls fall down when you drop them

If **any one** of the individual statements does not work, the generalization is invalid. (This generalization became problematic with the invention of the helium balloon.)

Scientists tend to work **empirically**, meaning they start with the specific facts and work their way back to the generalization. Generalizations are valued in science because they bring order to apparently disconnected facts, and that order in turn suggests underlying theories.

Mathematicians also spend a great deal of time looking for generalizations. When you have an "algebraic generalization" you have one **algebraic** fact that allows you to assume many **numerical** facts as examples. Consider, for instance, the first two functions in the function game.

1. Double the number, then add six.

2. Add three to the number, then double.

These are very different "recipes." However, their inclusion in the function game is a bit unfair, because here comes the generalization—**these two functions will always give the same answer.** Whether the input is positive or negative, integer or fraction, small or large, these two functions will mimic each other perfectly. We can express this generalization in words.

#### Example 2

**Generalization**: If you plug a number into the function **double and add six**, and plug the same number into the function **add three and double**, the two operations will give the same answer.

Specific facts, or examples:

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- If you double -5 and add six; or, if you add -5 to 3 and then double; you end up with the same answer.
- If you double 13 and add six; or, if you add 13 to 3 and then double; you end up with the same answer.

There is literally an infinite number of specific claims that fit this pattern. We don't need to prove or test each of these claims individually: once we have proven the generalization, we know that **all** these facts must be true.

We can express this same generalization pictorially by showing two "function machines" that always do the same thing.



Table 1

	2(x+3)	
	<b>344</b>	
$-5 \rightarrow$		$\rightarrow 2(-5) + 6 = -4$
$0 \rightarrow$	—	$\rightarrow 2(0) + 6 = 6$
$13 \rightarrow$		$\rightarrow 2(13) + 6 = 32$

Table 2

But the most common way to express this generalization is algebraically, by asserting that these two functions **equal** each other.

$$2x + 6 = 2(x + 3) \tag{1}$$

Many beginning Algebra II students will recognize this as the distributive property. Given 2(x+3) they can correctly turn it into 2x + 6. But they often fail to realize what this equality means—that given the same input, the two functions will always yield the same output.

Example 3 Generalization: 2x + 6 = 2(x + 3)Specific facts, or examples:

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- $(2 \times -5) + 6 = 2 \times (-5 + 3)$
- (2×0)+6=2×(0+3)
  (2×13)+6=2×(13+3)

 $(2 \times 10) + 0 = 2 \times (10 + 0)$ 

It's worth stopping for a moment here to think about the = symbol. Whenever it is used, = indicates that two things are the same. However, the following two equations use the = in very different ways.

$$2x^2 + 5x = 3$$
 (2)

$$\frac{2x^2 - 18}{x+3} = 2x - 6\tag{3}$$

In the first equation, the = challenges you to solve for x. "Find all the x values that make this equation true." The answers in this case are  $x = \frac{1}{2}$  and x = -3. If you plug in either of these two x-values, you get a true equation; for any other x-value, you get a false equation.

The second equation cannot be solved for x; the = sign in this case is asserting an equality that is true for **any**x-value. Let's try a few.

Example 4

Generalization:  $\frac{2x^2-18}{x+3} = 2x-6$ Specific facts, or examples:

x = 3	$\frac{2(3)^2 - 18}{(3) + 3} = \frac{18 - 18}{6} = 0$	$2(3) - 6 = 0 \checkmark$
x = -2	$\frac{2(-2)^2 - 18}{(-2) + 3} = \frac{8 - 18}{1} = -10$	$2(-2) - 6 = -10 \checkmark$
x = 0	$\frac{2(0)^2 - 18}{(0) + 3} = \frac{0 - 18}{3} = -6$	$2(0) - 6 = -6 \checkmark$
$x = \frac{1}{2}$	$\frac{2\left(\frac{1}{2}\right)^2 - 18}{\left(\frac{1}{2}\right) + 3} = \frac{\frac{1}{2} - 18}{\frac{7}{2}} = \left(\frac{-35}{2}\right)\left(\frac{2}{7}\right) = -5$	$2\left(\frac{1}{2}\right) - 6 = -5  \checkmark$

#### Table 3

With a calculator, you can attempt more difficult values such as x = -26 or  $x = \pi$ ; in every case, the two formulas will give the same answer. When we assert that two very different functions will always produce the same answers, we are making a very powerful generalization.

**Exception**: x = -3 is outside the domain of one of these two functions. In this important sense, the two functions are not in fact equal. Take a moment to make sure you understand why this is true!

Such generalizations are very important because they allow us to simplify.

Suppose that you were told "I am going to give you a hundred numbers. For each number I give you, square it, then double the answer, then subtract eighteen, then divide by the original number plus three." This kind of operation comes up all the time. But you would be quite relieved to discover that you can

accomplish the same task by simply doubling each number and subtracting 6! The generalization in this case is  $\frac{2x^2-18}{x+3} = 2x - 6$ ; you will be creating exactly this sort of generalization in the chapter on Rational Expressions.