

FUNCTION CONCEPTS – INVERSE FUNCTIONS*

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Abstract

This module describes what inverse functions are and how they can be used.

Let's go back to Alice, who makes \$100/day. We know how to answer questions such as "After 3 days, how much money has she made?" We use the function $m(t) = 100t$.

But suppose I want to ask the reverse question: "If Alice has made \$300, how many hours has she worked?" This is the job of an inverse function. It gives the same relationship, but reverses the dependent and independent variables. $t(m) = m/100$. Given any amount of money, divide it by 100 to find how many days she has worked.

If a function answers the question: "Alice worked this long, how much money has she made?" **then its inverse answers the question:** "Alice made this much money, how long did she work?"

If a function answers the question: "I have this many spoons, how much do they weigh?" **then its inverse answers the question:** "My spoons weigh this much, how many do I have?"

If a function answers the question: "How many hours of music fit on 12 CDs?" **then its inverse answers the question:** "How many CDs do you need for 3 hours of music?"

1 How do you recognize an inverse function?

Let's look at the two functions above:

$$m(t) = 100t \tag{1}$$

$$t(m) = m/100 \tag{2}$$

Mathematically, you can recognize these as inverse functions because they **reverse the inputs and the outputs**.

| |
|--|
| $3 \rightarrow m(t) = 100t \rightarrow 300$ |
| $300 \rightarrow t(m) = m/100 \rightarrow 3$ |
| ✓ Inverse functions |

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Table 1

Of course, this makes logical sense. The first line above says that “If Alice works 3 hours, she makes \$300.” The second line says “If Alice made \$300, she worked 3 hours.” It’s the same statement, made in two different ways.

But this “reversal” property gives us a way to test any two functions to see if they are inverses. For instance, consider the two functions:

$$f(x) = 3x + 7 \quad (3)$$

$$g(x) = \frac{1}{3}x - 7 \quad (4)$$

They **look** like inverses, don’t they? But let’s test and find out.

| |
|--|
| $2 \rightarrow 3x + 7 \rightarrow 13$ |
| $13 \rightarrow \frac{3}{x} - 7 \rightarrow \frac{13}{3} - 7 \rightarrow -\frac{8}{3}$ |
| × Not inverse functions |

Table 2

The first function turns a 2 into a 13. But the second function does **not** turn 13 into 2. So these are not inverses.

On the other hand, consider:

$$f(x) = 3x + 7 \quad (5)$$

$$g(x) = \frac{1}{3}(x - 7) \quad (6)$$

Let’s run our test of inverses on these two functions.

| |
|---|
| $2 \rightarrow 3x + 7 \rightarrow 13$ |
| $13 \rightarrow \frac{1}{3}(x - 7) \rightarrow 2$ |
| ✓ Inverse functions |

Table 3

So we can see that these functions do, in fact, reverse each other: they are inverses.

A common example is the Celsius-to-Fahrenheit conversion:

$$F(C) = \left(\frac{9}{5}\right)C + 32 \quad (7)$$

$$C(F) = \left(\frac{5}{9}\right)(F - 32) \quad (8)$$

where C is the Celsius temperature and F the Fahrenheit. If you plug $100^\circ C$ into the first equation, you find that it is $212^\circ F$. If you ask the second equation about $212^\circ F$, it of course converts that back into $100^\circ C$.

2 The notation and definition of an inverse function

The notation for the inverse function of $f(x)$ is $f^{-1}(x)$. This notation can cause considerable confusion, because it **looks like** an exponent, but it isn't. $f^{-1}(x)$ simply means "the inverse function of $f(x)$." It is defined formally by the fact that if you plug any number x into one function, and then plug the result into the other function, you get back where you started. (Take a moment to convince yourself that this is the same definition I gave above more informally.) We can represent this as a composition function by saying that $f(f^{-1}(x)) = x$.

Definition 1: Inverse Function

$f^{-1}(x)$ is defined as the **inverse function** of $f(x)$ if it consistently reverses the $f(x)$ process. That is, if $f(x)$ turns a into b , then $f^{-1}(x)$ must turn b into a . More concisely and formally, $f^{-1}(x)$ is the inverse function of $f(x)$ if $f(f^{-1}(x)) = x$.

3 Finding an inverse function

In examples above, we saw that if $f(x) = 3x + 7$, then $f^{-1}(x) = \frac{1}{3}(x - 7)$. We also saw that the function $\frac{1}{3}x - 7$, which may have looked just as likely, did **not** work as an inverse function. So in general, given a function, how do you find its inverse function?

Remember that an inverse function reverses the inputs and outputs. When we graph functions, we always represent the incoming number as x and the outgoing number as y . So to find the inverse function, **switch the x and y values, and then solve for y** .

Example 1: Building and Testing an Inverse Function

- Find the inverse function of $f(x) = \frac{2x-3}{5}$
 - Write the function as $y = \frac{2x-3}{5}$
 - Switch the x and y variables. $x = \frac{2y-3}{5}$
 - Solve for y . $5x = 2y - 3$. $5x + 3 = 2y$. $\frac{5x+3}{2} = y$. So $f^{-1}(x) = \frac{5x+3}{2}$.
- Test to make sure this solution fills the definition of an inverse function.
 - Pick a number, and plug it into the original function. $9 \rightarrow f(x) \rightarrow 3$.
 - See if the inverse function reverses this process. $3 \rightarrow f^{-1}(x) \rightarrow 9$. ✓ It worked!

Were you surprised by the answer? At first glance, it seems that the numbers in the original function (the 2, 3, and 5) have been rearranged almost at random.

But with more thought, the solution becomes very intuitive. The original function $f(x)$ described the following process: **double a number, then subtract 3, then divide by 5**. To reverse this process, we need to reverse each step in order: **multiply by 5, then add 3, then divide by 2**. This is just what the inverse function does.

4 Some functions have no inverse function

Some functions have no inverse function. The reason is the rule of consistency.

For instance, consider the function $y = x^2$. This function takes both 3 and -3 and turns them into 9. No problem: a function is allowed to turn different **inputs** into the same **output**. However, what does that say about the inverse of this particular function? In order to fulfill the requirement of an inverse function, it would have to take 9, and turn it into both 3 and -3—which is the one and only thing that functions are **not** allowed to do. Hence, the inverse of this function would not be a function at all!

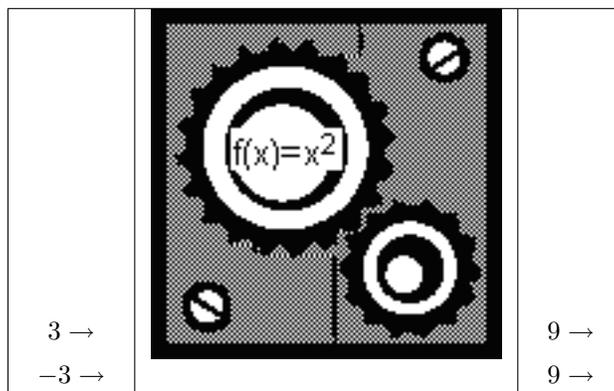


Table 4: If 3 goes in, 9 comes out. If -3 goes in, 9 also comes out. No problem:

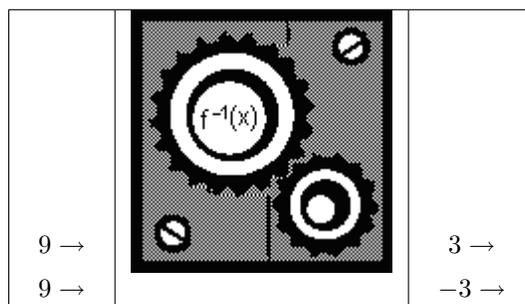


Table 5: But its inverse would have to turn 9 into both 3 and -3. No function can do this, so there is no inverse.

In general, any function that turns multiple inputs into the same output, does not have an inverse function.

What does that mean in the real world? If we can convert Fahrenheit to Celsius, we must be able to convert Celsius to Fahrenheit. If we can ask “How much money did Alice make in 3 days?” we must surely be able to ask “How long did it take Alice to make \$500?” When would you have a function that cannot be inverted?

Let’s go back to this example:

Recall the example that was used earlier: “Max threw a ball. The height of the ball depends on how many seconds it has been in the air.” The two variables here are h (the height of the ball) and t (the number of seconds it has been in the air). The function $h(t)$ enables us to answer questions such as “After 3 seconds, where is the ball?”

The inverse question would be “At what time was the ball 10 feet in the air?” The problem with that question is, it may well have **two answers!**

| The ball is here... | ...after this much time has elapsed |
|---------------------|-------------------------------------|
| 10 ft | 2 seconds (*on the way up) |
| 10 ft | 5 seconds (*on the way back down) |

Table 6

So what does that mean? Does it mean we can't ask that question? Of course not. We can ask that question, and we can expect to mathematically find the answer, or answers—and we will do so in the quadratic chapter. However, it does mean that **time is not a function of height** because such a “function” would not be consistent: one question would produce multiple answers.