

# LOGARITHM CONCEPTS – PROPERTIES OF LOGARITHMS\*

Kenny M. Felder

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## Abstract

This module contains some of the properties of logarithms and how they can be used for manipulation.

Just as there are three fundamental laws of exponents, there are three fundamental laws of logarithms.

$$\log_x(ab) = \log_x a + \log_x b \quad (1)$$

$$\log_x \frac{a}{b} = \log_x a - \log_x b \quad (2)$$

$$\log_x(a^b) = b \log_x a \quad (3)$$

As always, these **algebraic generalizations** hold for any  $a$ ,  $b$ , and  $x$ .

### Example 1: Properties of Logarithms

1. **Suppose you are given these two facts:**

$$\log_4 5 = 1.16$$

$$\log_4 10 = 1.66$$

2. **Then we can use the laws of logarithms to conclude that:**

$$\log_4(50) = \log_4 5 + \log_4 10 = 2.82$$

$$\log_4(2) = \log_4 10 - \log_4 5 = 0.5$$

$$\log_4(100,000) = 5 \log_4 10 = 8.3$$

NOTE: **All three of these results can be found quickly, and without a calculator. Note that the second result could also be figured out directly, since  $4^{\frac{1}{2}} = 2$ .**

These properties of logarithms were very important historically, because they enabled pre-calculator mathematicians to perform **multiplication** (which is very time-consuming and error prone) by doing **addition** (which is faster and easier). These rules are still useful in simplifying complicated expressions and solving equations.

### Example 2: Solving an equation with the properties of logarithms

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$\log_2 x - \log_x(x-1) = 5$	<b>The problem</b>	
$\log_2\left(\frac{x}{x-1}\right) = 5$	<b>Second property of logarithms</b>	
$\frac{x}{x-1} = 2^5 = 32$	<b>Rewrite the log as an exponent. (2-to-what is? <math>\frac{x}{x-1}</math> 2-to-the-5!)</b>	
$x = 32(x-1)$	<b>Multiply. We now have an easy equation to solve.</b>	
$x = 32x - 32$		
$-31x = -32$		
$x = \frac{32}{31}$		

Table 1

## 1 Proving the Properties of Logarithms

If you understand what an exponent is, you can very quickly see why the three rules of exponents work. But why do logarithms have these three properties?

As you work through the text, you will demonstrate these rules intuitively, by viewing the logarithm as a **counter**. ( $\log_2 8$  asks “**how many** 2s do I need to multiply, in order to get 8?”) However, these rules can also be rigorously proven, using the laws of exponents as our starting place.

Proving the First Law of Logarithms, $\log_x(ab) = \log_x a + \log_x b$	
$m = \log_x a$	<b>I’m just inventing <math>m</math> to represent this log</b>
$x^m = a$	<b>Rewriting the above expression as an exponent. (<math>\log_x a</math> asks “<math>x</math> to what power is <math>a</math>?” And the equation answers: “<math>x</math> to the <math>m</math> is <math>a</math>.”)</b>
$n = \log_x b$	<b>Similarly, <math>n</math> will represent the other log.</b>
$x^n = b$	
$\log_x(ab) = \log_x(x^m x^n)$	<b>Replacing <math>a</math> and <math>b</math> based on the previous equations</b>
$= \log_x(x^{m+n})$	<b>This is the key step! It uses the first law of exponents. Thus you can see that the properties of logarithms come directly from the laws of exponents.</b>
<i>continued on next page</i>	

$= m + n$	$= \log_x(x^{m+n})$ asks the question: “ $x$ to what power is $x^{m+n}$ ?” Looked at this way, the answer is obviously $(m + n)$ . Hence, you can see how the logarithm and exponential functions cancel each other out, as inverse functions must.
$= \log_x a + \log_x b$	Replacing $m$ and $n$ with what they were originally defined as. Hence, we have proven what we set out to prove.

Table 2

To test your understanding, try proving the second law of logarithms: the proof is very similar to the first. For the third law, you need invent only one variable,  $m = \log_x a$ . In each case, you will rely on a different one of the three rules of exponents, showing how each exponent law corresponds to one of the logarithms laws.