Radical Concepts – Simplifying Radicals*

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Abstract

This module covers techniques for the simplification of radicals.

1 Simplifying Radicals

The property $\sqrt{\text{ab}}= \sqrt{a}\sqrt{b}$ can be used to simplify radicals. The key is to break the number inside the root into two factors, **one of which is a perfect square**.

Example 1: Simplifying a Radical

$\sqrt{\mathbf{75}}$	
$= \sqrt{25 \bullet 3}$	**because 25•3 is 75, and 25 is a perfect square**
$= \sqrt{25}\sqrt{3}$	**because** $\sqrt{\mathbf{ab}} = \sqrt{a}\sqrt{b}$
$=5 \sqrt{3}$	**because** $\sqrt{25}=\mathbf{5}$

Table 1

So we conclude that $\sqrt{75}=5\sqrt{3}$. You can confirm this on your calculator (both are approximately 8.66).

We rewrote 75 as $25 \bullet 3$ because 25 is a perfect square. We could, of course, also rewrite 75 as $5 \bullet 15$, but—although correct—that would not help us simplify, because neither number is a perfect square.

Example 2: Simplifying a Radical in Two Steps

*Version 1.3: Nov 5, 2011 2:36 pm -0500

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$\sqrt{180}$	
$= \sqrt{9 \bullet 20}$	**because** $9 \bullet 20$ **is 180, and 9 is a perfect square**
$= \sqrt{9}\ \sqrt{20}$	**because** $\sqrt{\mathbf{ab}} = \sqrt{a}\ \sqrt{b}$
$= 3\ \sqrt{20}$	**So far, so good. But wait! We're not done!**
$= 3\ \sqrt{4 \bullet 5}$	**There's another perfect square to pull out!**
$= 3\ \sqrt{4}\ \sqrt{5}$	
$= 3\,(2)\ \sqrt{5}$	
$= 6\ \sqrt{5}$	**Now we're done.**

Table 2

The moral of this second example is that **after** you simplify, you should always look to see if you can simplify **again** .

A secondary moral is, try to pull out the biggest perfect square you can. We could have jumped straight to the answer if we had begun by rewriting 180 as $36 \bullet 5$.

This sort of simplification can sometimes allow you to **combine** radical terms, as in this example:

Example 3: Combining Radicals

$\sqrt{\mathbf{75}}\ \ --\ \ \sqrt{\mathbf{12}}$	
$= 5\ \ \sqrt{3}\ \ --2\ \ \sqrt{3}$	**We found earlier that** $\sqrt{\mathbf{75}} = 5\ \sqrt{3}$ **. Use the same method to confirm that** $\sqrt{\mathbf{12}} = 2\ \sqrt{3}$ **.**
$= 3\ \ \sqrt{3}$	**5 of anything minus 2 of that same thing is 3 of it, right?**

Table 3

That last step may take a bit of thought. It can only be used when the radical is the same. Hence, $\sqrt{2} + \sqrt{3}$ **cannot be simplified at all. We were able to simplify** $\sqrt{\mathbf{75}} - \sqrt{\mathbf{12}}$ **only by making the radical in both cases the same .**

So why does $5\sqrt{3}--2\sqrt{3}= 3\sqrt{3}$? It may be simplest to think about verbally: 5 of these things, minus 2 of the same things, is 3 of them. But you can look at it more formally as a factoring problem, if you see a common factor of $\sqrt{3}$.

$5\sqrt{3}--2\sqrt{3}=\sqrt{3}(5--2)=\sqrt{3}(3)$.

Of course, the process is exactly the same if variable are involved instead of just numbers!

Example 4: Combining Radicals with Variables

$x^{\frac{3}{2}} + x^{\frac{5}{2}}$	
$= x^3 + x^5$	**Remember the definition of fractional exponents!**
$= \sqrt{x^2 * x} + \sqrt{x^4 * x}$	**As always, we simplify radicals by factoring them inside the root...**
$\sqrt{x^2} * \sqrt{x} + \sqrt{x^4} * \sqrt{x}$	**and then breaking them up...**
$= x\sqrt{x} + x^2\sqrt{x}$	**and then taking square roots outside!**
$= \left(x^2 + x\right)\sqrt{x}$	**Now that the radical is the same, we can combine.**

Table 4

1.1 Rationalizing the Denominator

It is always possible to express a fraction with no square roots in the denominator.

Is it always desirable? Some texts are religious about this point: "You should never have a square root in the denominator." I have absolutely no idea why. To me, $\frac{1}{\sqrt{2}}$ looks simpler than $\frac{\sqrt{2}}{2}$; I see no overwhelming reason for forbidding the first or preferring the second.

However, there are times when it is useful to remove the radicals from the denominator: for instance, when adding fractions. The trick for doing this is based on the basic rule of fractions: **if you multiply the top and bottom of a fraction by the same number, the fraction is unchanged.** This rule enables us to say, for instance, that $\frac{2}{3}$ is exactly the same number as $\frac{2 \cdot 3}{3 \cdot 3} = \frac{6}{9}$.

In a case like $\frac{1}{\sqrt{2}}$, therefore, you can multiply the top and bottom by $\sqrt{2}$.

$\frac{1}{\sqrt{2}} = \frac{1*2}{\sqrt{2}*\sqrt{2}} = \frac{\sqrt{2}}{2}$

What about a more complicated case, such as $\frac{\sqrt{12}}{1+\sqrt{3}}$? You might think we could simplify this by multiplying the top and bottom by $(1+\sqrt{3})$, but that doesn't work: the bottom turns into $(1+3)^2 = 1 + 2\sqrt{3} + 3$, which is at least as ugly as what we had before.

The correct trick for getting rid of $(1+\sqrt{3})$ is to multiply it by $(1 - -\sqrt{3})$. These two expressions, identical except for the replacement of a+ by a−, are known as **conjugates**. What happens when we multiply them? We don't need to use FOIL if we remember that

$(x+y)(x-y) = x^2 - y^2$

Using this formula, we see that

$\left(1+\sqrt{3}\right)\left(1-\sqrt{3}\right) = 1^2 - \left(\sqrt{3}\right)^2 = 1 - 3 = -2$

So the square root does indeed go away. We can use this to simplify the original expression as follows.

Example 5: Rationalizing Using the Conjugate of the Denominator

$\frac{\sqrt{12}}{1+\sqrt{3}} = \frac{\sqrt{12}\left(1-\sqrt{3}\right)}{\left(1+\sqrt{3}\right)\left(1-\sqrt{3}\right)} = \frac{\sqrt{12}-\sqrt{36}}{1-3} = \frac{2\sqrt{3}-6}{-2} = -\sqrt{3} + 3$

As always, you may want to check this on your calculator. Both the original and the simplified expression are approximately 1.268.

Of course, the process is the same when variables are involved.

Example 6: Rationalizing with Variables

$\frac{1}{x-\sqrt{x}} = \frac{1\left(x+\sqrt{x}\right)}{\left(x-\sqrt{x}\right)\left(x+\sqrt{x}\right)} = \frac{x+\sqrt{x}}{x^2-x}$

Once again, we multiplied the top and the bottom by the **conjugate of the denominator**: that is, we replaced a− with a+. The formula $(x+a)(x-a) = x^2 - a^2$ enabled us to quickly multiply the terms on the bottom, and eliminated the square roots in the denominator.