Connexions module: m18294

## Matrix Concepts - The Inverse Matrix\*

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## Abstract

This module introduces the inverse matrix and its properties.

We have seen that the number 1 plays a special role in multiplication, because 1x = x.

The inverse of a number is defined as the number that multiplies by that number to give 1: b is the inverse of a if ab = 1. Hence, the inverse of 3 is  $\frac{1}{3}$ ; the inverse of  $\frac{-5}{8} = \frac{-8}{5}$ . Every number except 0 has an inverse.

By analogy, the **inverse of a matrix** multiplies by that matrix to give the identity matrix.

## **Definition of Inverse Matrix**

The **inverse of matrix** [A], designated as  $[A]^{-1}$ , is defined by the property:  $[A][A]^{-1} = [A]^{-1}[A] = [I]$ The superscript -1 is being used here in a similar way to its use in functions. Recall that  $f^{--1}(x)$  does not designate an exponent of any kind, but instead, an inverse function. In the same way,  $[A]^{--1}$  does not denote an exponent, but an inverse matrix.

Note that, just as in the definition of the identity matrix, this definition requires commutativity—the multiplication must work the same in either order.

Note also that only square matrices can have an inverse. Why? The definition of an inverse matrix is based on the identity matrix [I], and we already said that only square matrices even have an identity!

How do you find an inverse matrix? The method comes directly from the definition, with a little algebra.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
 as the inverse that we are looking for, by asserting that it fills the definition of

an inverse matrix: when you multiply this mystery matrix by our original matrix, you get [I]. When we solve for the four variables a, b, c, and d, we will have found our inverse matrix.

$$\begin{bmatrix} 3a+4c & 3b+4d \\ 5a+6c & 5b+6d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
 Do the multiplication. (You should check this step for

yourself, it's great practice. For instance, you start by multiplying first row x first column, and you get 3a+4c.)

5a+6c=0 5b+6d=1 Remember what it means for two matrices 3a + 4c = 13b + 4d = 0to be equal: every element in the left must equal its corresponding element on the right. So, for these two matrices to equal each other, all four of these equations must hold.

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and you get 3a+4c.)

a=-3 b=2  $c=2\frac{1}{2}$   $d=-1\frac{1}{2}$  Solve the first two equations for a and c by using either elimination or substitution. Solve the second two equations for b and d by using either elimination or substitution. (The steps are not shown here.)

So the inverse is:  $\begin{bmatrix} -3 & 2 \\ 2\frac{1}{2} & -1\frac{1}{2} \end{bmatrix}$  Having found the four variables, we have found the inverse.  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  as the inverse that we are looking for, by asserting that it fills the definition

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		$\begin{bmatrix} a & b \\ \end{bmatrix}$ as the in-
Example: Finding an Inverse Matrix		$\begin{bmatrix} c & d \end{bmatrix}$ as the in-
Find the inverse of 3 4	The problem	
5 6	The problem	
$\left  \begin{array}{c c} 3 & 4 \\ 5 & 6 \end{array} \right  \left  \begin{array}{cc} a & b \\ c & d \end{array} \right  = \left  \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right $	This is the key step. It establishes	
$\begin{bmatrix} 5 & 6 \end{bmatrix} \begin{bmatrix} c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$	]   Ima is one neg seep. It escapiones	
b		
c	d	

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Table 1

Did it work? Let's find out.

Testing our Inverse Matrix	
	continued on next page

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$     \begin{bmatrix}       -3 & 2 \\       2\frac{1}{2} & -1\frac{1}{2}     \end{bmatrix}     \begin{bmatrix}       3 & 4 \\       5 & 6     \end{bmatrix}     \stackrel{?}{=}     \begin{bmatrix}       1 & 0 \\       0 & 1     \end{bmatrix} $	The definition of an inverse matrix: if we have indeed found an inverse, then when we multiply it by the original matrix, we should get [I].
$ \left[ \begin{array}{ccc} (-3)(3) + (2)(5) & (-3)(4) + (2)(6) \\ (2\frac{1}{2})(3) + (-1\frac{1}{2})(5) & (2\frac{1}{2})(4) + (-1\frac{1}{2})(6) \end{array} \right] \stackrel{?}{=} $	Do the multiplication.
$\begin{bmatrix} -9 + 10 & -12 + 12 \\ 7\frac{1}{2} - 7\frac{1}{2} & 10 - 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	It works!

Table 2

Note that, to fully test it, we would have to try the multiplication in both orders. Why? Because, in general, changing the order of a matrix multiplication changes the answer; but the definition of an inverse matrix specifies that it must work both ways! Only one order was shown above, so technically, we have only half-tested this inverse.

This process does not have to be memorized: it should make logical sense. Everything we have learned about matrices should make logical sense, except for the very arbitrary-looking definition of matrix multiplication.