

INTRODUCTION TO CONCISE SIGNAL MODELS*

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Abstract

This collection reviews fundamental concepts underlying the use of concise models for signal processing. Topics are presented from a geometric perspective and include low-dimensional linear, sparse, and manifold-based signal models, approximation, compression, dimensionality reduction, and Compressed Sensing.

1 Overview

In characterizing a given problem in signal processing, one is often able to specify a model for the signals to be processed. This model may distinguish (either statistically or deterministically) classes of interesting signals from uninteresting ones, typical signals from anomalies, information from noise, etc.

Very commonly, models in signal processing deal with some notion of structure, constraint, or conciseness. Roughly speaking, one often believes that a signal has “few degrees of freedom” relative to the size of the signal. This notion of conciseness is a very powerful assumption, and it suggests the potential for dramatic gains via algorithms that capture and exploit the true underlying structure of the signal.

In these modules, we survey three common examples of concise models: linear models, sparse nonlinear models, and manifold-based models. In each case, we discuss an important phenomenon: the conciseness of the model corresponds to a low-dimensional geometric structure along which the signals of interest tend to cluster. This low-dimensional geometry again has important implications in the understanding and the development of efficient algorithms for signal processing.

We discuss this low-dimensional geometry in several contexts, including projecting a signal onto the model class (i.e., forming a concise approximation to a signal), encoding such an approximation (i.e., data compression), and reducing the dimensionality of signals and data sets. We conclude with an important and emerging application area known as Compressed Sensing (CS), which is a novel method for data acquisition that relies on concise models and builds upon strong geometric principles. We discuss CS in its traditional, sparsity-based context and also discuss extensions of CS to other concise models such as manifolds.

2 General Mathematical Preliminaries

2.1 Signal notation

We will treat signals as real- or complex-valued functions having domains that are either discrete (and finite) or continuous (and either compact or infinite). Each of these assumptions will be made clear as needed. As

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a general rule, however, we will use x to denote a discrete signal in \mathbb{R}^N and f to denote a function over a continuous domain \mathcal{D} . We also commonly refer to these as discrete- or continuous-**time** signals, though the domain need not actually be temporal in nature.

2.2 L_p and ℓ_p norms

As measures for signal energy, fidelity, or sparsity, we will employ the L_p and ℓ_p norms. For continuous-time functions, the L_p norm is defined as

$$\|f\|_{L_p(\mathcal{D})} = \left(\int_{\mathcal{D}} |f|^p\right)^{1/p}, \quad p \in (0, \infty), \quad (1)$$

and for discrete-time functions, the ℓ_p norm is defined as

$$\|x\|_{\ell_p} = \begin{cases} \left(\sum_{i=1}^N |x(i)|^p\right)^{1/p}, & p \in (0, \infty), \\ \max_{i=1, \dots, N} |x(i)|, & p = \infty, \\ \sum_{i=1}^N \mathbf{1}_{x(i) \neq 0}, & p = 0, \end{cases} \quad (2)$$

where $\mathbf{1}$ denotes the indicator function. (While we often refer to these measures as “norms,” they actually do not meet the technical criteria for norms when $p < 1$.)

2.3 Linear algebra

Let A be a real-valued $M \times N$ matrix. We denote the **nullspace** of A as $\mathcal{N}(A)$ (note that $\mathcal{N}(A)$ is a linear subspace of \mathbb{R}^N), and we denote the **transpose** of A as A^T .

We call A an **orthoprojector** from \mathbb{R}^N to \mathbb{R}^M if it has orthonormal rows. From such a matrix we call $A^T A$ the corresponding **orthogonal projection operator** onto the M -dimensional subspace of \mathbb{R}^N spanned by the rows of A .