Sets and Counting*

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Abstract

This chapter covers principles of sets and counting. After completing this chapter students should be able to: use set theory and venn diagrams to solve counting problems; use the multiplication axiom to solve counting problems; use permutations to solve counting problems; use combinations to solve counting problems; and use the binomial theorem to expand x+y^n.

1 Chapter Overview

In this chapter, you will learn to:

1. Use set theory and Venn diagrams to solve counting problems.
2. Use the Multiplication Axiom to solve counting problems.
3. Use Permutations to solve counting problems.
4. Use Combinations to solve counting problems.
5. Use the Binomial Theorem to expand $(x+y)^n$.

2 Sets

In this section, we will familiarize ourselves with set operations and notations, so that we can apply these concepts to both counting and probability problems. We begin by defining some terms.

A **set** is a collection of objects, and its members are called the **elements** of the set. We name the set by using capital letters, and enclose its members in braces. Suppose we need to list the members of the chess club. We use the following set notation.

$$C = \{\text{Ken, Bob, Tran, Shanti, Eric}\} \tag{1}$$

A set that has no members is called an **empty set**. The empty set is denoted by the symbol Ø.

Two sets are **equal** if they have the same elements.

A set A is a **subset** of a set B if every member of A is also a member of B.

Suppose $C = \{\text{Al, Bob, Chris, David, Ed}\}$ and $A = \{\text{Bob, David}\}$. Then A is a subset of C, written as $A \subseteq C$.

Every set is a subset of itself, and the empty set is a subset of every set.

Union Of Two Sets

Let A and B be two sets, then the union of A and B, written as $A \cup B$, is the set of all elements that are either in A or in B, or in both A and B.

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Intersection Of Two Sets
Let A and B be two sets, then the intersection of A and B, written as $A \cap B$, is the set of all elements that are common to both sets A and B.

A **universal set** U is the set consisting of all elements under consideration.

Complement of a Set
Let A be any set, then the complement of set A, written as $\overset{\Psi}{A}$, is the set consisting of elements in the universal set U that are not in A.

Disjoint Sets
Two sets A and B are called disjoint sets if their intersection is an empty set.

Example 1
List all the subsets of the set of primary colors {red, yellow, blue}.

Solution
The subsets are $\emptyset$, {red}, {yellow}, {blue}, {red, yellow}, {red, blue}, {yellow, blue}, {red, yellow, blue}

Note that the empty set is a subset of every set, and a set is a subset of itself.

Example 2
Let F = {Aikman, Jackson, Rice, Sanders, Young}, and B = {Griffey, Jackson, Sanders, Thomas}. Find the intersection of the sets F and B.

Solution
The intersection of the two sets is the set whose elements belong to both sets. Therefore,

$$F \cap B = \{\text{Jackson, Sanders}\} \quad (2)$$

Example 3
Find the union of the sets F and B given as follows.

$$F = \{\text{Aikman, Jackson, Rice, Sanders, Young}\} B = \{\text{Griffey, Jackson, Sanders, Thomas}\} \quad (3)$$

Solution
The union of two sets is the set whose elements are either in A or in B or in both A and B. Therefore

$$F \cup B = \{\text{Aikman, Griffey, Jackson, Rice, Sanders, Thomas, Young}\} \quad (4)$$

Observe that when writing the union of two sets, the repetitions are avoided.

Example 4
Let the universal set U = {red, orange, yellow, green, blue, indigo, violet}, and P = {red, yellow, blue}. Find the complement of P.

Solution

The complement of a set P is the set consisting of elements in the universal set U that are not in P. Therefore,

$$\overset{\Psi}{P} = \{\text{orange, green, indigo, violet}\}(5)$$

To achieve a better understanding, let us suppose that the universal set U represents the colors of the spectrum, and P the primary colors, then $\overset{\Psi}{P}$ represents those colors of the spectrum that are not primary colors.

Example 5

Let $U = \{\text{red, orange, yellow, green, blue, indigo, violet}\}$, $P = \{\text{red, yellow, blue}\}$, $Q = \{\text{red, green}\}$, and $R = \{\text{orange, green, indigo}\}$. Find $\overline{P \cup Q} \cap \overset{\Psi}{R}$.

Solution

We do the problems in steps.

$$\begin{aligned} P \cup Q &= \{\text{red, yellow, blue, green}\} \\ \overline{P \cup Q} &= \{\text{orange, indigo, violet}\} \\ \overset{\Psi}{R} &= \{\text{red, yellow, blue, violet}\} \\ \overline{P \cup Q} \cap \overset{\Psi}{R} &= \{\text{violet}\} \end{aligned} \quad (6)$$

We now use Venn diagrams to illustrate the relations between sets. In the late 1800s, an English logician named John Venn developed a method to represent relationship between sets. He represented these relationships using diagrams, which are now known as Venn diagrams. A Venn diagram represents a set as the interior of a circle. Often two or more circles are enclosed in a rectangle where the rectangle represents the universal set. To visualize an intersection or union of a set is easy. In this section, we will mainly use Venn diagrams to sort various populations and count objects.

Example 6

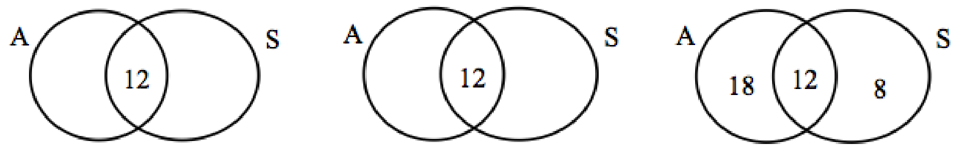
Suppose a survey of car enthusiasts showed that over a certain time period, 30 drove cars with automatic transmissions, 20 drove cars with standard transmissions, and 12 drove cars of both types. If every one in the survey drove cars with one of these transmissions, how many people participated in the survey?

Solution

We will use Venn diagrams to solve this problem.

Let the set A represent those car enthusiasts who drove cars with automatic transmissions, and set S represent the car enthusiasts who drove the cars with standard transmissions. Now we use Venn diagrams to sort out the information given in this problem.

Since 12 people drove both cars, we place the number 12 in the region common to both sets.



(a) (b) (c)

Figure 1

Because 30 people drove cars with automatic transmissions, the circle A must contain 30 elements. This means $x + 12 = 30$, or $x = 18$. Similarly, since 20 people drove cars with standard transmissions, the circle B must contain 20 elements, or $y + 12 = 20$ which in turn makes $y = 8$.

Now that all the information is sorted out, it is easy to read from the diagram that 18 people drove cars with automatic transmissions only, 12 people drove both types of cars, and 8 drove cars with standard transmissions only. Therefore, $18 + 12 + 8 = 38$ people took part in the survey.

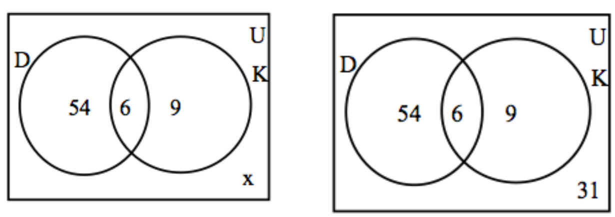
Example 7

A survey of 100 people in California indicates that 60 people have visited Disneyland, 15 have visited Knott's Berry Farm, and 6 have visited both. How many people have visited neither place?

Solution

The problem is similar to the one in Example 6.

Let the set D represent the people who have visited Disneyland, and K the set of people who have visited Knott's Berry Farm.



(a) (b)

Figure 2

We fill the three regions associated with the sets D and K in the same manner as before. Since 100 people participated in the survey, the rectangle representing the universal set U must contain 100 objects. Let x represent those people in the universal set that are neither in the set D nor in K. This means $54 + 6 + 9 + x = 100$, or $x = 31$.

Therefore, there are 31 people in the survey who have visited neither place.

Example 8

A survey of 100 exercise conscious people resulted in the following information:

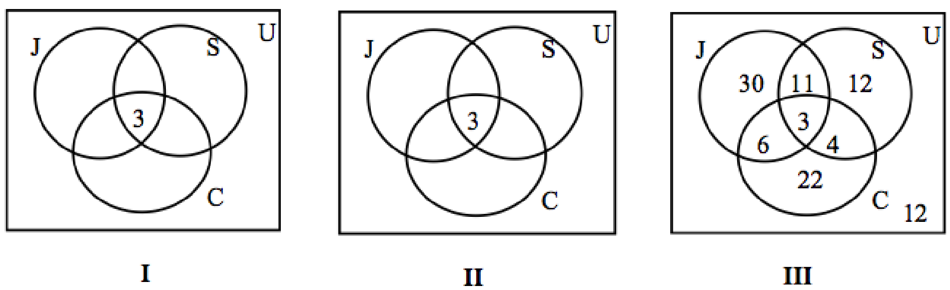
- 50 jog, 30 swim, and 35 cycle
- 14 jog and swim
- 7 swim and cycle
- 9 jog and cycle
- 3 people take part in all three activities

a. How many jog but do not swim or cycle?
b. How many take part in only one of the activities?
c. How many do not take part in any of these activities?

Solution

Let J represent the set of people who jog, S the set of people who swim, and C who cycle.

In using Venn diagrams, our ultimate aim is to assign a number to each region. We always begin by first assigning the number to the innermost region and then working our way out.



(a) (b) (c)

Figure 3

We place a 3 in the innermost region of Figure 3(a) because it represents the number of people who participate in all three activities. Next we compute x, y and z.

Since 14 people jog and swim, $x + 3 = 14$, or $x = 11$.

The fact that 9 people jog and cycle results in $y + 3 = 9$, or $y = 6$.

Since 7 people swim and cycle, $z + 3 = 7$, or $z = 4$.

This information is depicted in Figure 3(b).

Now we proceed to find the unknowns m, n and p.

Since 50 people jog, $m + 11 + 6 + 3 = 50$, or $m = 30$.

Thirty people swim, therefore, $n + 11 + 4 + 3 = 30$, or $n = 12$.

Thirty five people cycle, therefore, $p + 6 + 4 + 3 = 35$, or $p = 22$.

By adding all the entries in all three sets, we get a sum of 88. Since 100 people were surveyed, the number inside the universal set but outside of all three sets is 100 − 88, or 12.

In Figure 3(c), the information is sorted out, and the questions can readily be answered.

a. The number of people who jog but do not swim or cycle is 30.
b. The number who take part in only one of these activities is $30 + 12 + 22 = 64$.
c. The number of people who do not take part in any of these activities is 12.

3 Tree Diagrams and the Multiplication Axiom

In this chapter, we are trying to develop counting techniques that will be used in the here[1] to study probability. One of the most fundamental of such techniques is called the Multiplication Axiom. Before we introduce the multiplication axiom, we first look at some examples.

Example 9

If a woman has two blouses and three skirts, how many different outfits consisting of a blouse and a skirt can she wear?

Solution

Suppose we call the blouses b_1 and b_2, and skirts s_1, s_2, and s_3.

We can have the following six outfits.

$$b_1s_1, b_1s_2, b_1s_3, b_2s_1, b_2s_2, b_2s_3 \quad (7)$$

Alternatively, we can draw a tree diagram:

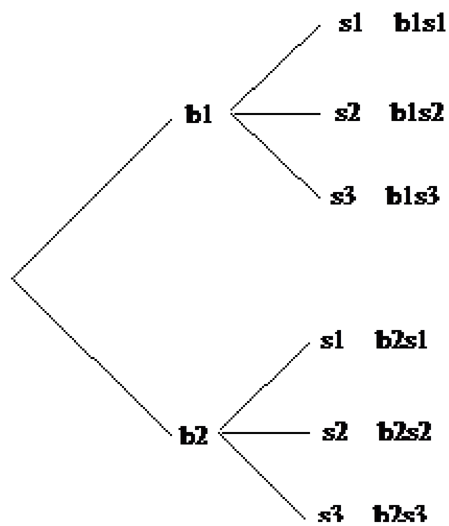


Figure 4

The tree diagram gives us all six possibilities. The method involves two steps. First the woman chooses a blouse. She has two choices: blouse one or blouse two. If she chooses blouse one, she has three skirts to match it with; skirt one, skirt two, or skirt three. Similarly if she chooses blouse two, she can match it with each of the three skirts, again. The tree diagram helps us visualize these possibilities.

The reader should note that the process involves two steps. For the first step of choosing a blouse, there are two choices, and for each choice of a blouse, there are three choices of choosing a skirt. So altogether there are $2 \cdot 3 = 6$ possibilities.

If, in the above example, we add the shoes to the outfit, we have the following problem.

[1] "Probability" <http://cnx.org/content/m18907/latest/>

Example 10

If a woman has two blouses, three skirts, and two pumps, how many different outfits consisting of a blouse, a skirt, and a pair of pumps can she wear?

Solution

Suppose we call the blouses b_1 and b_2, the skirts s_1, s_2, and s_3, and the pumps p_1, and p_2.

The following tree diagram results.

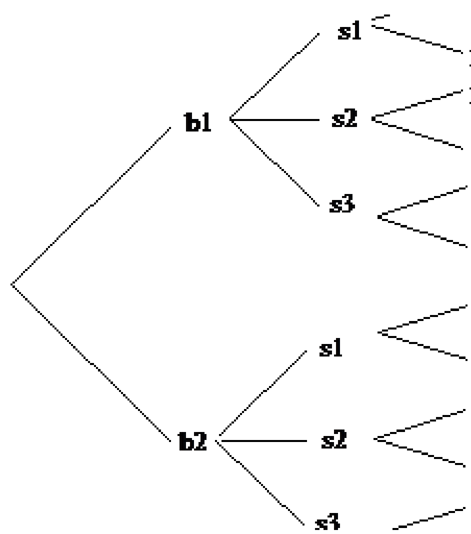


Figure 5

We count the number of branches in the tree, and see that there are 12 different possibilities. This time the method involves three steps. First, the woman chooses a blouse. She has two choices: blouse one or blouse two. Now suppose she chooses blouse one. This takes us to step two of the process which consists of choosing a skirt. She has three choices for a skirt, and let us suppose she chooses skirt two. Now that she has chosen a blouse and a skirt, we have moved to the third step of choosing a pair of pumps. Since she has two pairs of pumps, she has two choices for the last step. Let us suppose she chooses pumps two. She has chosen the outfit consisting of blouse one, skirt two, and pumps two, or $b_1 s_2 p_2$. By looking at the different branches on the tree, one can easily see the other possibilities.

The important thing to observe here, again, is that this is a three step process. There are two choices for the first step of choosing a blouse. For each choice of a blouse, there are three choices of choosing a skirt, and for each combination of a blouse and a skirt, there are two choices of selecting a pair of pumps. All in all, we have $2 \cdot 3 \cdot 2 = 12$ different possibilities.

The tree diagrams help us visualize the different possibilities, but they are not practical when the possibilities are numerous. Besides, we are mostly interested in finding the number of elements in the set and not the actual possibilities. But once the problem is envisioned, we can solve it without a tree diagram. The two examples we just solved may have given us a clue to do just that.

Let us now try to solve Example 10 without a tree diagram. Recall that the problem involved three steps: choosing a blouse, choosing a skirt, and choosing a pair of pumps. The number of ways of choosing each are listed below.

The Number of ways of choosing a blouse	The number of ways of choosing a skirt	The number of ways of choosing pumps
2	3	2

Table 1

By multiplying these three numbers we get 12, which is what we got when we did the problem using a tree diagram.

The procedure we just employed is called the multiplication axiom.

11: THE MULTIPLICATION AXIOM

If a task can be done in m ways, and a second task can be done in n ways, then the operation involving the first task followed by the second can be performed in $m \cdot n$ ways.

The general multiplication axiom is not limited to just two tasks and can be used for any number of tasks.

Example 12

A truck license plate consists of a letter followed by four digits. How many such license plates are possible?

Solution

Since there are 26 letters and 10 digits, we have the following choices for each.

Letter	Digit	Digit	Digit	Digit
26	10	10	10	10

Table 2

Therefore, the number of possible license plates is $26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 260,000$.

Example 13

In how many different ways can a 3-question true-false test be answered?

Solution

Since there are two choices for each question, we have

Question 1	Question 2	Question 3
2	2	2

Table 3

Applying the multiplication axiom, we get $2 \cdot 2 \cdot 2 = 8$ different ways.

We list all eight possibilities below.

TTT, TTF, TFT, TFF, FTT, FTF, FFT, FFF

The reader should note that the first letter in each possibility is the answer corresponding to the first question, the second letter corresponds to the answer to the second question and so on. For example, TFF, says that the answer to the first question is given as true, and the answers to the second and third questions false.

Example 14

In how many different ways can four people be seated in a row?

Solution

Suppose we put four chairs in a row, and proceed to put four people in these seats.

There are four choices for the first chair we choose. Once a person sits down in that chair, there are only three choices for the second chair, and so on. We list as shown below.

4	3	2	1

Table 4

So there are altogether $4 \cdot 3 \cdot 2 \cdot 1 = 24$ different ways.

Example 15

How many three-letter word sequences can be formed using the letters $\{A, B, C\}$ if no letter is to be repeated?

Solution

The problem is very similar to Example 14.

Imagine a child having three building blocks labeled A, B, and C. Suppose he puts these blocks on top of each other to make word sequences. For the first letter he has three choices, namely A, B, or C. Let us suppose he chooses the first letter to be a B, then for the second block which must go on top of the first, he has only two choices: A or C. And for the last letter he has only one choice. We list the choices below.

3	2	1

Table 5

Therefore, 6 different word sequences can be formed.

Finally, we'd like to illustrate this with a tree diagram.

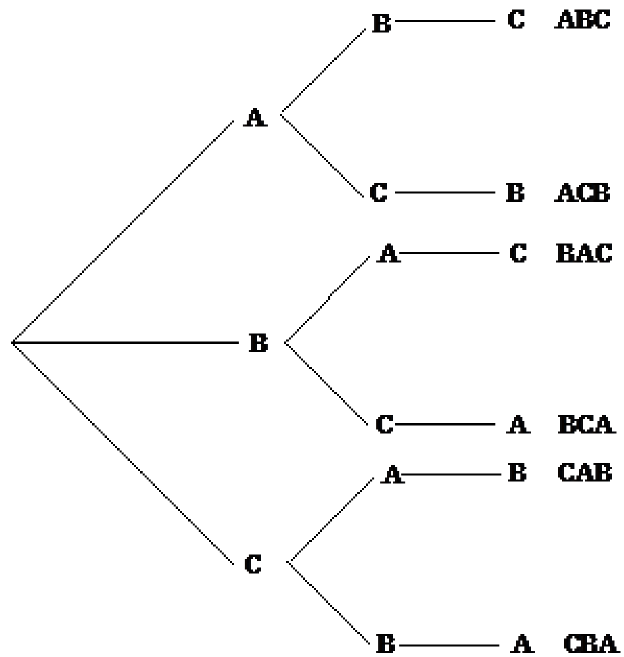


Figure 6

All six possibilities are displayed in the tree diagram.

4 Permutations

In Example 15, we were asked to find the word sequences formed by using the letters $\{A, B, C\}$ if no letter is to be repeated. The tree diagram gave us the following six arrangements.

ABC, ACB, BAC, BCA, CAB, and CBA,

Arrangements like these, where order is important and no element is repeated, are called permutations.

16: Permutations

A permutation of a set of elements is an ordered arrangement where each element is used once.

Example 17

How many three-letter word sequences can be formed using the letters $\{A, B, C, D\}$?

Solution

There are four choices for the first letter of our word, three choices for the second letter, and two choices for the third.

4	3	2

Table 6

Applying the multiplication axiom, we get $4 \cdot 3 \cdot 2 = 24$ different arrangements.

Example 18

How many permutations of the letters of the word ARTICLE have consonants in the first and last positions?

Solution

In the word ARTICLE, there are 4 consonants.

Since the first letter must be a consonant, we have four choices for the first position, and once we use up a consonant, there are only three consonants left for the last spot. We show as follows:

4						3

Table 7

Since there are no more restrictions, we can go ahead and make the choices for the rest of the positions.

So far we have used up 2 letters, therefore, five remain. So for the next position there are five choices, for the position after that there are four choices, and so on. We get

4	5	4	3	2	1	3

Table 8

So the total permutations are $4 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 = 1440$.

Example 19

Given five letters $\{A, B, C, D, E\}$. Find the following:

a. The number of four-letter word sequences.
b. The number of three-letter word sequences.
c. The number of two-letter word sequences.

Solution

The problem is easily solved by the multiplication axiom, and answers are as follows:

a. a. The number of four-letter word sequences is $5 \cdot 4 \cdot 3 \cdot 2 = 120$.
b. b. The number of three-letter word sequences is $5 \cdot 4 \cdot 3 = 60$.
c. c. The number of two-letter word sequences is $5 \cdot 4 = 20$.

We often encounter situations where we have a set of n objects and we are selecting r objects to form permutations. We refer to this as **permutations of n objects taken r at a time**, and we write it as n**Pr**.

Therefore, Example 19 can also be answered as listed below.

a. The number of four-letter word sequences is $5P4 = 120$.

b. The number of three-letter word sequences is $5P3 = 60$.
c. The number of two-letter word sequences is $5P2 = 20$.

Before we give a formula for nPr, we'd like to introduce a symbol that we will use a great deal in this as well as in here[2].

20: Factorial
$n! = n(n-1)(n-2)(n-3)\cdots 3 \cdot 2 \cdot 1$.
Where n is a natural number.

$$0! = 1 \qquad (8)$$

Now we define n**Pr**.

21: The Number of Permutations of n Objects Taken r at a Time
$n\text{Pr} = n(n-1)(n-2)(n-3)\cdots(n-r+1)$, or
$n\text{Pr} = \frac{n!}{(n-r)!}$
Where n and r are natural numbers.

The reader should become familiar with both formulas and should feel comfortable in applying either.

Example 22
Compute the following using both formulas.

a. $6P3$
b. $7P2$

Solution
We will identify n and r in each case and solve using the formulas provided.

a. $6P3 = 6 \cdot 5 \cdot 4 = 120$, alternately $6P3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 120$
b. $7P2 = 7 \cdot 6 = 42$, or $7P2 = \frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 42$

Next we consider some more permutation problems to get further insight into these concepts.

Example 23
In how many different ways can 4 people be seated in a straight line if two of them insist on sitting next to each other?

Solution
Let us suppose we have four people A, B, C, and D. Further suppose that A and B want to sit together. For the sake of argument, we tie A and B together and treat them as one person.

The four people are $\boxed{\text{AB}}$ CD . Since $\boxed{\text{AB}}$ is treated as one person, we have the following possible arrangements.

$$\boxed{\text{AB}}\ \text{CD}\ , \boxed{\text{AB}}\ \text{DC}\ ,\ C\ \boxed{\text{AB}}\ D\ ,\ D\ \boxed{\text{AB}}\ C\ ,\ \text{CD}\ \boxed{\text{AB}},\ \text{DC}\ \boxed{\text{AB}} \qquad (9)$$

Note that there are six more such permutations because A and B could also be tied in the order BA. And they are

$$\boxed{\text{BA}}\ \text{CD}\ , \boxed{\text{BA}}\ \text{DC}\ ,\ C\ \boxed{\text{BA}}\ D\ ,\ D\ \boxed{\text{BA}}\ C\ ,\ \text{CD}\ \boxed{\text{BA}},\ \text{DC}\ \boxed{\text{BA}} \qquad (10)$$

[2]"Probability" <http://cnx.org/content/m18907/latest/>

So altogether there are 12 different permutations.

Let us now do the problem using the multiplication axiom.

After we tie two of the people together and treat them as one person, we can say we have only three people. The multiplication axiom tells us that three people can be seated in 3! ways. Since two people can be tied together 2! ways, there are $3!2! = 12$ different arrangements.

Example 24

You have 4 math books and 5 history books to put on a shelf that has 5 slots. In how many ways can the books be shelved if the first three slots are filled with math books and the next two slots are filled with history books?

Solution

We first do the problem using the multiplication axiom.

Since the math books go in the first three slots, there are 4 choices for the first slot, 3 for the second and 2 for the third. The fourth slot requires a history book, and has five choices. Once that choice is made, there are 4 history books left, and therefore, 4 choices for the last slot. The choices are shown below.

4	3	2	5	4

Table 9

Therefore, the number of permutations are $4 \cdot 3 \cdot 2 \cdot 5 \cdot 4 = 480$.

Alternately, we can see that $4 \cdot 3 \cdot 2$ is really same as $4P3$, and $5 \cdot 4$ is $5P2$.

So the answer can be written as $(P3)\ \ (5P2) = 480$.

Clearly, this makes sense. For every permutation of three math books placed in the first three slots, there are $5P2$ permutations of history books that can be placed in the last two slots. Hence the multiplication axiom applies, and we have the answer $(4P3)\ \ (5P2)$.

We summarize.

1. **Permutations** A permutation of a set of elements is an ordered arrangement where each element is used once.
2. **Factorial** $n! = n(n-1)(n-2)(n-3)\cdots 3 \cdot 2 \cdot 1$.
   Where n is a natural number.
$$0! = 1 \qquad (11)$$
3. **Permutations of n Objects Taken r at a Time** $n\text{Pr} = n(n-1)(n-2)(n-3)\cdots(n-r+1)$, or $n\text{Pr} = \frac{n!}{(n-r)!}$
   Where n and r are natural numbers.

5 Circular Permutations and Permutations with Similar Elements

5.1 Section Overview

In this section we will address the following two problems.

1. In how many different ways can five people be seated in a circle?
2. In how many different ways can the letters of the word MISSISSIPPI be arranged?

The first problem comes under the category of Circular Permutations, and the second under Permutations with Similar Elements.

Circular Permutations

Suppose we have three people named A, B, and C. We have already determined that they can be seated in a straight line in 3! or 6 ways. Our next problem is to see how many ways these people can be seated in a circle. We draw a diagram.

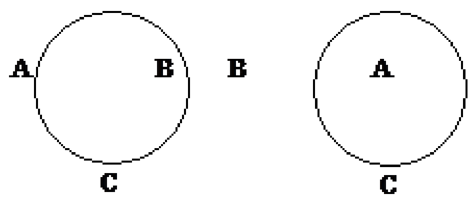


Figure 7

It happens that there are only two ways we can seat three people in a circle. This kind of permutation is called a circular permutation. In such cases, no matter where the first person sits, the permutation is not affected. Each person can shift as many places as they like, and the permutation will not be changed. Imagine the people on a merry-go-round; the rotation of the permutation does not generate a new permutation. So in circular permutations, the first person is considered a place holder, and where he sits does not matter.

25: Circular Permutations

The number of permutations of n elements in a circle is $(n-1)!$

Example 26

In how many different ways can five people be seated at a circular table?

Solution

We have already determined that the first person is just a place holder. Therefore, there is only one choice for the first spot. We have

1	4	3	2	1

Table 10

So the answer is 24.

Example 27

In how many ways can four couples be seated at a round table if the men and women want to sit alternately?

Solution

We again emphasize that the first person can sit anywhere without affecting the permutation.

So there is only one choice for the first spot. Suppose a man sat down first. The chair next to it must belong to a woman, and there are 4 choices. The next chair belongs to a man, so there are three choices and so on. We list the choices below.

1	4	3	3	2	2	1	1

Table 11

So the answer is 144.

Now we address the second problem.

Permutations with Similar Elements

Let us determine the number of distinguishable permutations of the letters ELEMENT.

Suppose we make all the letters different by labeling the letters as follows.

$$E_1\text{LE}_2\text{ME}_3\text{NT} \quad (12)$$

Since all the letters are now different, there are 7! different permutations.

Let us now look at one such permutation, say

$$\text{LE}_1\text{ME}_2\text{NE}_3T \quad (13)$$

Suppose we form new permutations from this arrangement by only moving the E's. Clearly, there are 3! or 6 such arrangements. We list them below.

$$\text{LE}_1\text{ME}_2\text{NE}_3T \quad (14)$$

$$\text{LE}_1\text{ME}_3\text{NE}_2T \quad (15)$$

$$\text{LE}_2\text{ME}_1\text{NE}_3T \quad (16)$$

$$\text{LE}_3\text{ME}_3\text{NE}_1T \quad (17)$$

$$\text{LE}_3\text{ME}_2\text{NE}_1T \quad (18)$$

$$\text{LE}_3\text{ME}_1\text{NE}_2T \quad (19)$$

Because the E's are not different, there is only one arrangement LEMENET and not six. This is true for every permutation.

Let us suppose there are n different permutations of the letters ELEMENT.

Then there are $n \cdot 3!$ permutations of the letters $E_1\text{LE}_2\text{ME}_3\text{NT}$.

But we know there are 7! permutations of the letters $E_1\text{LE}_2\text{ME}_3\text{NT}$.

Therefore, $n \cdot 3! = 7!$

Or $n = \frac{7!}{3!}$.

This gives us the method we are looking for.

28: Permutations with Similar Elements

The number of permutations of n elements taken n at a time, with r_1 elements of one kind, r_2 elements of another kind, and so on, is

$$\frac{n!}{r_1!r_2!...r_k!} \tag{20}$$

Example 29

Find the number of different permutations of the letters of the word MISSISSIPPI.

Solution

The word MISSISSIPPI has 11 letters. If the letters were all different there would have been 11! different permutations. But MISSISSIPPI has 4 S's, 4 I's, and 2 P's that are alike.

So the answer is $\frac{11!}{4!4!2!}$

Which equals 34,650.

Example 30

If a coin is tossed six times, how many different outcomes consisting of 4 heads and 2 tails are there?

Solution

Again, we have permutations with similar elements.

We are looking for permutations for the letters HHHHTT.

The answer is $\frac{6!}{4!2!} = 15$.

Example 31

In how many different ways can 4 nickels, 3 dimes, and 2 quarters be arranged in a row?

Solution

Assuming that all nickels are similar, all dimes are similar, and all quarters are similar, we have permutations with similar elements. Therefore, the answer is

$$\frac{9!}{4!3!2!} = 1260 \tag{21}$$

Example 32

A stock broker wants to assign 20 new clients equally to 4 of its salespeople. In how many different ways can this be done?

Solution

This means that each sales person gets 5 clients. The problem can be thought of as an ordered partitions problem. In that case, using the formula we get

$$\frac{20!}{5!5!5!5!} = 11,732,745,024 \tag{22}$$

We summarize.

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1. **Circular Permutations** The number of permutations of n elements in a circle is

$$(n-1)! \tag{23}$$

2. **Permutations with Similar Elements** The number of permutations of n elements taken n at a time, with r_1 elements of one kind, r_2 elements of another kind, and so on, such that $n = r_1 + r_2 + \cdots + r_k$ is
$\frac{n!}{r_1! r_2! \cdots r_k!}$
This is also referred to as **ordered partitions**.

6 Combinations

Suppose we have a set of three letters $\{A, B, C\}$, and we are asked to make two-letter word sequences. We have the following six permutations.

AB BA BC CB AC CA

Now suppose we have a group of three people $\{A, B, C\}$ as Al, Bob, and Chris, respectively, and we are asked to form committees of two people each. This time we have only three committees, namely,

AB BC AC

When forming committees, the order is not important, because the committee that has Al and Bob is no different than the committee that has Bob and Al. As a result, we have only three committees and not six.

Forming word sequences is an example of permutations, while forming committees is an example of **combinations** – the topic of this section.

Permutations are those arrangements where order is important, while combinations are those arrangements where order is not significant. From now on, this is how we will tell permutations and combinations apart.

In Example 32, there were six permutations, but only three combinations.

Just as the symbol nPr represents the number of permutations of n objects taken r at a time, nCr represents the number of combinations of n objects taken r at a time.

So in Example 32, $3P2 = 6$, and $3C2 = 3$.

Our next goal is to determine the relationship between the number of combinations and the number of permutations in a given situation.

In Example 32, if we knew that there were three combinations, we could have found the number of permutations by multiplying this number by 2!. That is because each combination consists of two letters, and that makes 2! permutations.

Example 34

Given the set of letters $\{A, B, C, D\}$. Write the number of combinations of three letters, and then from these combinations determine the number of permutations.

Solution

We have the following four combinations.

ABC BCD CDA BDA

Since every combination has three letters, there are 3! permutations for every combination. We list them below.

ABC BCD CDA BDA
ACB BDC CAD BAD
BAC CDB DAC DAB

BCA CBD DCA DBA
CAB DCB ACD ADB
CBA DBC ADC ABD
The number of permutations are 3! times the number of combinations. That is

$$4p3 = 3! \cdot 4C3 \tag{24}$$

or $4C3 = \frac{4P3}{3!}$
In general, nCr $= \frac{n\text{Pr}}{r!}$
Since $n\text{Pr} = \frac{n!}{(n-r)!}$
We have, nCr $= \frac{n!}{(n-r)!r!}$

Summarizing,

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1. **Combinations** A combination of a set of elements is an arrangement where each element is used once, and order is not important.
2. **The Number of Combinations of n Objects Taken r at a Time** nCr $= \frac{n!}{(n-r)!r!}$
Where n and r are natural numbers.

Example 36
Compute:

a. $5C3$
b. $7C3$.

Solution
We use the above formula.

$$5C3 = \frac{5!}{(5-3)!3!} = \frac{5!}{2!3!} = 10 \tag{25}$$

$$7C3 = \frac{7!}{(7-3)!3!} = \frac{7!}{4!3!} = 35. \tag{26}$$

Example 37
In how many different ways can a student select to answer five questions from a test that has seven questions, if the order of the selection is not important?

Solution
Since the order is not important, it is a combination problem, and the answer is
$7C5 = 21$.

Example 38
How many line segments can be drawn by connecting any two of the six points that lie on the circumference of a circle?

Solution

Since the line that goes from point A to point B is same as the one that goes from B to A, this is a combination problem.

It is a combination of 6 objects taken 2 at a time. Therefore, the answer is

$$6C2 = \frac{6!}{4!2!} = 15 \tag{27}$$

Example 39

There are ten people at a party. If they all shake hands, how many hand-shakes are possible?

Solution

Note that between any two people there is only one hand shake. Therefore, we have

$$10\text{C2} = 45 \quad \text{hand-shakes.} \tag{28}$$

Example 40

The shopping area of a town is in the shape of square that is 5 blocks by 5 blocks. How many different routes can a taxi driver take to go from one corner of the shopping area to the opposite cater-corner?

Solution

Let us suppose the taxi driver drives from the point A, the lower left hand corner, to the point B, the upper right hand corner as shown in the figure below.

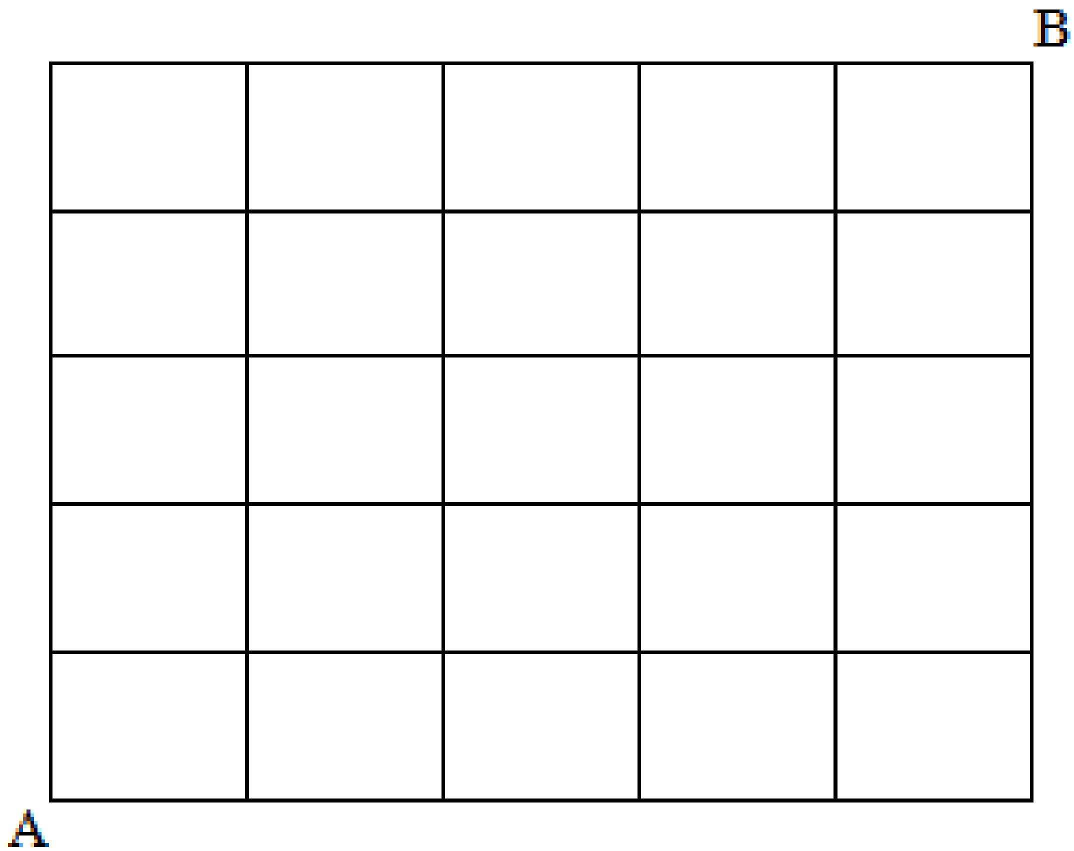

Figure 8

To reach his destination, he has to travel ten blocks; five horizontal, and five vertical. So if out of the ten blocks he chooses any five horizontal, the other five will have to be the vertical blocks, and vice versa. Therefore, all he has to do is to choose 5 out of ten.

The answer is 10C5, or 252.

Alternately, the problem can be solved by permutations with similar elements.

The taxi driver's route consists of five horizontal and five vertical blocks. If we call a horizontal block H, and a vertical block a V, then one possible route may be as follows.

$$\text{HHHHHVVVVV} \quad (29)$$

Clearly there are $\frac{10!}{5!5!} = 252$ permutations.

Further note that by definition $10C5 = \frac{10!}{5!5!}$.

Example 41

If a coin is tossed six times, in how many ways can it fall four heads and two tails?

Solution

First we solve this problem using Section 6 (Combinations) technique–permutations with similar elements.

We need 4 heads and 2 tails, that is

$$\text{HHHHTT} \tag{30}$$

There are $\frac{6!}{4!2!} = 15$ permutations.

Now we solve this problem using combinations.

Suppose we have six spots to put the coins on. If we choose any four spots for heads, the other two will automatically be tails. So the problem is simply

$6C4 = 15.$

Incidentally, we could have easily chosen the two tails, instead. In that case, we would have gotten

$6C2 = 15.$

Further observe that by definition

$$6C4 = \frac{6!}{2!4!} \tag{31}$$

and $6C2 = \frac{6!}{4!2!}$

Which implies

$$6C4 = 6C2. \tag{32}$$

7 Combinations: Involving Several Sets

So far we have solved the basic combination problem of r objects chosen from n different objects. Now we will consider certain variations of this problem.

Example 42

How many five-people committees consisting of 2 men and 3 women can be chosen from a total of 4 men and 4 women?

Solution

We list 4 men and 4 women as follows:

$$M_1 M_2 M_3 M_4 W_1 W_2 W_3 W_4 \tag{33}$$

Since we want 5-people committees consisting of 2 men and 3 women, we'll first form all possible two-man committees and all possible three-woman committees. Clearly there are 4C2 = 6 two-man committees, and 4C3 = 4 three-woman committees, we list them as follows:

2-Man Committees	3-Woman Committees
M_1M_2	$W_1W_2W_3$
M_1M_3	$W_1W_2W_4$
M_1M_4	$W_1W_3W_4$
M_2M_3	$W_2W_3W_4$
M_2M_4	
M_3M_4	

Table 12

For every 2-man committee there are four 3-woman committees that can be chosen to make a 5-person committee. If we choose M_1M_2 as our 2-man committee, then we can choose any of $W_1W_2W_3$, $W_1W_2W_4$, $W_1W_3W_4$, or $W_2W_3W_4$ as our 3-woman committees. As a result, we get

$$\boxed{M_1M_2}, W_1W_2W_3 \boxed{M_1M_2}, W_1W_2W_4 \boxed{M_1M_2}, W_1W_3W_4 \boxed{M_1M_2}, W_2W_3W_4 \quad (34)$$

Similarly, if we choose M_1M_3 as our 2-man committee, then, again, we can choose any of $W_1W_2W_3$, $W_1W_2W_4$, $W_1W_3W_4$, or $W_2W_3W_4$ as our 3-woman committees.

$$\boxed{M_1M_3}, W_1W_2W_3 \boxed{M_1M_3}, W_1W_2W_4 \boxed{M_1M_3}, W_1W_3W_4 \boxed{M_1M_3}, W_2W_3W_4 \quad (35)$$

And so on.

Since there are six 2-man committees, and for every 2-man committee there are four 3- woman committees, there are altogether $6 \cdot 4 = 24$ five-people committees.

In essence, we are applying the multiplication axiom to the different combinations.

Example 43

A high school club consists of 4 freshmen, 5 sophomores, 5 juniors, and 6 seniors. How many ways can a committee of 4 people be chosen that includes

a. One student from each class?
b. All juniors?
c. Two freshmen and 2 seniors?
d. No freshmen?
e. At least three seniors?

Solution

a. Applying the multiplication axiom to the combinations involved, we get

$$4C1 \cdot 5C1 \cdot 5C1 \cdot 6C1 = 600 \quad (36)$$

b. We are choosing all 4 members from the 5 juniors, and none from the others.

$$5C4 = 5 \quad (37)$$

c. $4C2 \cdot 6C2 = 90$

d. Since we don't want any freshmen on the committee, we need to choose all members from the remaining 16. That is

$$16\text{C}4 = 1820 \quad (38)$$

e. Of the 4 people on the committee, we want at least three seniors. This can be done in two ways. We could have three seniors, and one non-senior, or all four seniors.

$$6C3 \cdot 14\text{C}1 + 6C4 = 295 \quad (39)$$

Example 44
How many five-letter word sequences consisting of 2 vowels and 3 consonants can be formed from the letters of the word INTRODUCE?

Solution
First we select a group of five letters consisting of 2 vowels and 3 consonants. Since there are 4 vowels and 5 consonants, we have

$$4C2 \cdot 5C3 \quad (40)$$

Since our next task is to make word sequences out of these letters, we multiply these by 5!.
$4C2 \cdot 5C3 \cdot 5! = 7200$.

Example 45
A standard deck of playing cards has 52 cards consisting of 4 suits each with 13 cards. In how many different ways can a 5-card hand consisting of four cards of one suit and one of another be drawn?

Solution
We will do the problem using the following steps. Step 1. Select a suit. Step 2. Select four cards from this suit. Step 3. Select another suit. Step 4. Select a card from that suit.

Applying the multiplication axiom, we have

Ways of selecting a suit	Ways if selecting 4 cards from this suit	Ways if selecting the next suit	Ways of selecting a card from that suit
$4C1$	13C4	$3C1$	13C1

Table 13

$$4C1 \cdot 13C4 \cdot 3C1 \cdot 13C1 = 111,540. \quad (41)$$