Continuous Random Variables: The Uniform Distribution (modified R. Bloom)*

Roberta Bloom

Based on *Continuous Random Variables: The Uniform Distribution*† by

Susan Dean

Barbara Illowsky, Ph.D.

Abstract

This module examines the properties of the continuous Uniform probability distribution, which describes a set of continuous data for which all intervals of values having the same length are equally likely. This revision is based on the original module m16819 in the textbook collection Collaborative Statistics by S. Dean and Dr. B. Illowsky; the last example in the original module was replaced with a new example.

Example 1

Illustrate the uniform distribution. The data that follows are a random sample of 55 smiling times, in seconds, of an eight-week old baby.

10.4	19.6	18.8	13.9	17.8	16.8	21.6	17.9	12.5	11.1	4.9
12.8	14.8	22.8	20.0	15.9	16.3	13.4	17.1	14.5	19.0	22.8
1.3	0.7	8.9	11.9	10.9	7.3	5.9	3.7	17.9	19.2	9.8
5.8	6.9	2.6	5.8	21.7	11.8	3.4	2.1	4.5	6.3	10.7
8.9	9.4	9.4	7.6	10.0	3.3	6.7	7.8	11.6	13.8	18.6

Table 1

sample mean = 11.49 and sample standard deviation = 6.23

*Version 1.2: Nov 15, 2010 1:45 pm -0600
†http://cnx.org/content/m16819/1.11/

We will assume that the sample of smiling times is drawn from a population that follows a uniform distribution between 0 and 23 seconds. This means that any smiling time between 0 and 23 seconds is equally likely.

Let X = length of time, in seconds, of an eight-week old's smile.

The notation for the uniform distribution is

$X \sim U\,(\mathrm{a},b)$ where a = the lowest value and b = the highest value.

For this example, $X \sim U\,(0, 23)$. $0 < X < 23$.

Formulas for the theoretical mean and standard deviation are

$\mu = \frac{a+b}{2}$ and $\sigma = \sqrt{\frac{(b-a)^2}{12}}$

For this problem, the theoretical mean and standard deviation are

$\mu = \frac{0+23}{2} = 11.50$ seconds and $\sigma = \sqrt{\frac{(23-0)^2}{12}} = 6.64$ seconds

Notice that the theoretical mean and standard deviation are close to the sample mean and standard deviation.

To find $f\,(X)$: $f\,(X) = \frac{1}{23-0} = \frac{1}{23}$

Example 2

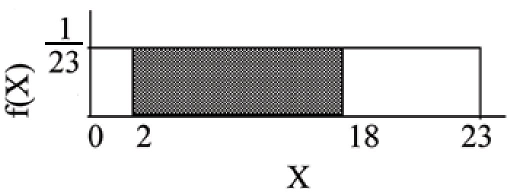
Problem 1

What is the probability that a randomly chosen eight-week old smiles between 2 and 18 seconds?

Solution

Find $P\,(2 < X < 18)$.

$P\,(2 < X < 18) = (\text{base})\,(\text{height}) = (18-2)\cdot\frac{1}{23} = \frac{16}{23}$.



Problem 2

Find the 90th percentile for an eight week old's smiling time.

Solution

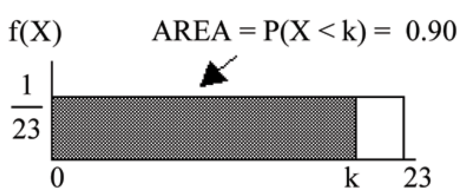
Ninety percent of the smiling times fall below the 90th percentile, k, so $P\,(X < k) = 0.90$

$P\,(X < k) = 0.90$

$(\text{base})\,(\text{height}) = 0.90$

$(k-0)\cdot\frac{1}{23} = 0.90$

$k = 23 \cdot 0.90 = 20.7$



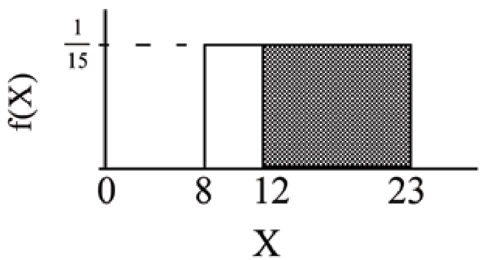
Problem 3: Problem 3 is OPTIONAL: Conditional Probability

Find the probability that a random eight week old smiles more than 12 seconds **KNOWING** that the baby smiles **MORE THAN 8 SECONDS**.

Solution

Find $P(X > 12|X > 8)$ There are two ways to do the problem. **For the first way**, use the fact that this is a **conditional** and changes the sample space. The graph illustrates the new sample space. You already know the baby smiled more than 8 seconds.

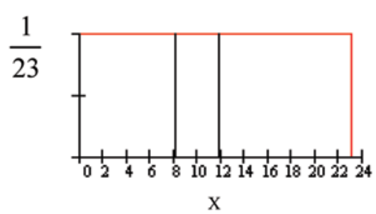
Write a new $f(X)$**:** $f(X) = \frac{1}{23-8} = \frac{1}{15}$
for $8 < X < 23$
$P(X > 12|X > 8) = (23 - 12) \cdot \frac{1}{15} = \frac{11}{15}$



For the second way, use the conditional formula from Chapter 3 with the original distribution $X \sim U(0, 23)$:

$P(A|B) = \frac{P(A \text{ AND } B)}{P(B)}$ For this problem, A is $(X > 12)$ and B is $(X > 8)$.

So, $P(X > 12|X > 8) = \frac{(X>12 \text{ AND } X>8)}{P(X>8)} = \frac{P(X>12)}{P(X>8)} = \frac{\frac{11}{23}}{\frac{15}{23}} = 0.733$



Example 3

Uniform: The amount of time, in minutes, that a person must wait for a bus is uniformly distributed between 0 and 15 minutes.

Problem 1

What is the probability that a person waits fewer than 12.5 minutes?

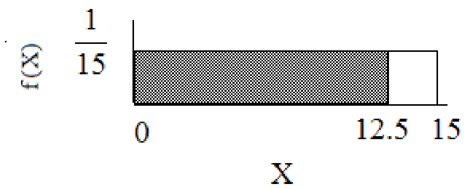
Solution

Let X = the number of minutes a person must wait for a bus. $a = 0$ and $b = 15$. $X \sim U(0, 15)$. Write the probability density function. $f(X) = \frac{1}{15-0} = \frac{1}{15}$ for $0 < X < 15$

Find $P(X < 12.5)$. Draw a graph.

$P(X < k) = (\text{base})(\text{height}) = (12.5 - 0) \cdot \frac{1}{15} = 0.8333$

The probability a person waits less than 12.5 minutes is 0.8333.



Problem 2

On the average, how long must a person wait?

Find the mean, μ, and the standard deviation, σ.

Solution

$\mu = \frac{a+b}{2} = \frac{15+0}{2} = 7.5$. On the average, a person must wait 7.5 minutes.

$\sigma = \sqrt{\frac{(b-a)^2}{12}} = \sqrt{\frac{(15-0)^2}{12}} = 4.3$. The Standard deviation is 4.3 minutes.

Problem 3

Ninety percent of the time, the time a person must wait falls below what value?

NOTE: This asks for the 90th percentile.

Solution

Find the 90th percentile. Draw a graph. Let k = the 90th percentile.

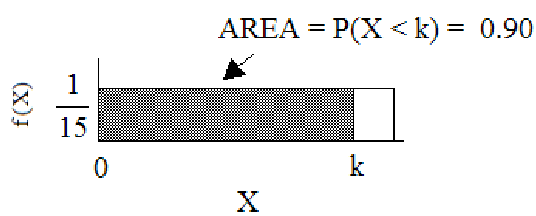
$P(X < k) = (\text{base})(\text{height}) = (k - 0) \cdot \left(\frac{1}{15}\right)$

$0.90 = k \cdot \frac{1}{15}$

k = (0.90) (15) = 13.5

k is sometimes called a critical value.

The 90th percentile is 13.5 minutes. Ninety percent of the time, a person must wait at most 13.5 minutes.



Example 4

Uniform: Ace Heating and Air Conditioning Service finds that the amount of time a repairman needs to fix a furnace is uniformly distributed between 1.5 and 4 hours. Let X = the time needed to fix a furnace. Then $X \sim U(1.5, 4)$.

1. Find the problem that a randomly selected furnace repair requires more than 2 hours.
2. Find the probability that a randomly selected furnace repair requires less than 3 hours.
3. Find the 30th percentile of furnace repair times.

4. The longest 25% of repair furnace repairs take at least how long? (In other words: Find the minimum time for the longest 25% of repair times.) What percentile does this represent?
5. Find the mean and standard deviation

Problem 1

Find the probability that a randomly selected furnace repair requires longer than 2 hours.

Solution

To find $f(X)$: $f(X) = \frac{1}{4-1.5} = \frac{1}{2.5}$ so $f(X)0.4$

P(X>2) = (base)(height) = $(4-2)(0.4) = 0.8$

Example 4 Figure 1

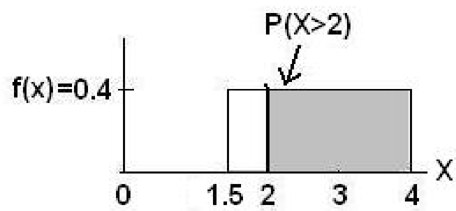


Figure 1: Uniform Distribution between 1.5 and 4 with shaded area between 2 and 4 representing the probability that the repair time X is greater than 2

Problem 2

Find the probability that a randomly selected furnace repair requires less than 3 hours. Describe how the graph differs from the graph in the first part of this example.

Solution

$P(X < 3) = \text{(base)(height)} = (3-1.5)(0.4) = 0.6$

The graph of the rectangle showing the entire distribution would remain the same. However the graph should be shaded between X=1.5 and X=3. Note that the shaded area starts at X=1.5 rather than at X=0; since X∼U(1.5,4), X can not be less than 1.5.

Example 4 Figure 2

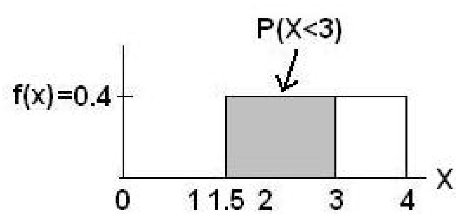


Figure 2: Uniform Distribution between 1.5 and 4 with shaded area between 1.5 and 3 representing the probability that the repair time X is less than 3

Problem 3
Find the 30th percentile of furnace repair times.

Solution

Example 4 Figure 3

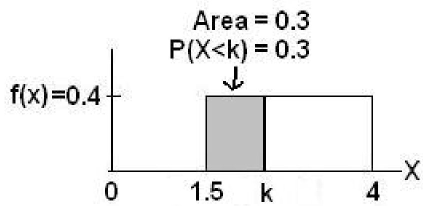


Figure 3: Uniform Distribution between 1.5 and 4 with an area of 0.30 shaded to the left, representing the shortest 30% of repair times.

$P(X < k) = 0.30$
$P(X < k) = (\text{base})(\text{height}) = (k - 1.5) \cdot (0.4)$

0.3 = (k − 1.5) (0.4) ; Solve to find k:
0.75 = k − 1.5 , obtained by dividing both sides by 0.4
k = 2.25 , obtained by adding 1.5 to both sides

The 30th percentile of repair times is 2.25 hours. 30% of repair times are 2.5 hours or less.

Problem 4
The **longest 25%** of furnace repair times take **at least** how long? (Find the minimum time for the longest 25% of repairs.)

Solution

Example 4 Figure 4

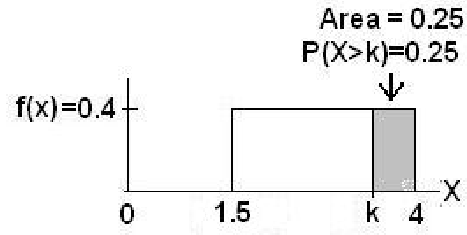


Figure 4: Uniform Distribution between 1.5 and 4 with an area of 0.25 shaded to the right representing the longest 25% of repair times.

$P(X > k) = 0.25$
$P(X > k) = (\text{base})(\text{height}) = (4 - k) \cdot (0.4)$

$\mathbf{0.25 = (4 - k)(0.4)}$; Solve for k:
$0.625 = 4 - \text{k}$, obtained by dividing both sides by 0.4
$-3.375 = -\text{k}$, obtained by subtracting 4 from both sides
k=3.375

The longest 25% of furnace repairs take at least 3.375 hours (3.375 hours or longer).

Note: Since 25% of repair times are 3.375 hours or longer, that means that 75% of repair times are 3.375 hours or less. 3.375 hours is the **75th percentile** of furnace repair times.

Problem 5
Find the mean and standard deviation

Solution
$\mu = \frac{a+b}{2}$ and $\sigma = \sqrt{\frac{(b-a)^2}{12}}$
$\mu = \frac{1.5+4}{2} = 2.75$ hours and $\sigma = \sqrt{\frac{(4-1.5)^2}{12}} = 0.7217$ hours

NOTE: See "Summary of the Uniform and Exponential Probability Distributions[1]" for a full summary.

Glossary

Definition 1: Conditional Probability
The likelihood that an event will occur given that another event has already occurred.

Definition 2: Uniform Distribution
Continuous random variable (RV) that appears to have equally likely outcomes over the domain, $a < x < b$. Often referred as **Rectangular distribution** because graph of its pdf has form of

[1] "Continuous Random Variables: Summary of The Uniform and Exponential Probability Distributions" <http://cnx.org/content/m16813/latest/>

rectangle. Notation: X~$U(a,b)$. The mean is $\mu = \frac{a+b}{2}$, and the variance is $\sigma^2 = \frac{(b-a)^2}{12}$, the probability density function is $f(x) = \frac{1}{b-a}, a \leq X \leq b$, and cumulative distribution is $P(X \leq x) = \frac{x-a}{b-a}$.