

CHANGE OF BASIS AND CHANGE TO SIGNAL*

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Abstract

One can look at the operation of a matrix times a vector as changing the basis set for the vector or as changing the vector with the same basis description.

1 Change of Basis

The operation given in can be viewed as \mathbf{x} being a signal vector and with \mathbf{b} being a vector whose entries are inner products of \mathbf{x} and the rows of \mathbf{A} . In other words, the elements of \mathbf{b} are the projection coefficients of \mathbf{x} onto the coordinates given by the rows of \mathbf{A} . The multiplication of a signal by this operator decomposes it and gives the coefficients of the decomposition.

An alternative view has \mathbf{x} being a set of weights so that \mathbf{b} is a weighted sum of the columns of \mathbf{A} . In other words, \mathbf{b} will lie in the space spanned by the columns of \mathbf{A} at a location determined by \mathbf{x} . This view is a composition of a signal from a set of weights which could have been obtained from a previous decomposition.

These two views of the operation as a decomposition of a signal or the recomposition of the signal to or from a different basis system are extremely valuable in signal analysis. The ideas of orthogonality, rank, adjoint, etc. are all important here. The dimensions of the domain and range of the operators may or may not be the same. The matrices may or may not be square and may or may not be of full rank [4].

A set of linearly independent vectors \mathbf{x}_n forms a basis for a vector space if every vector \mathbf{x} in the space can be uniquely written

$$\mathbf{x} = \sum_n a_n \mathbf{x}_n \quad (1)$$

and the dual basis vectors $\tilde{\mathbf{x}}_n$ allow a simple inner product to calculate the expansion coefficients as

$$a_n = \langle \mathbf{x}, \tilde{\mathbf{x}}_n \rangle = \mathbf{x}^T \tilde{\mathbf{x}}_n \quad (2)$$

(1) can be written as a matrix operation

$$\mathbf{F}\mathbf{a}=\mathbf{x} \quad (3)$$

where the columns of \mathbf{F} are the basis vectors and the vector \mathbf{a} has the expansion coefficients a_n as entries. Equation (2) can also be written as a matrix operation

$$\tilde{\mathbf{F}}\mathbf{x}=\mathbf{a} \quad (4)$$

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which has the dual basis vectors as rows of $\tilde{\mathbf{F}}$. From (3) and (4), we have

$$\mathbf{F}\tilde{\mathbf{F}}\mathbf{x}=\mathbf{x} \quad (5)$$

Since this is true for all \mathbf{x} ,

$$\mathbf{F}\tilde{\mathbf{F}}=\mathbf{I} \quad (6)$$

or

$$\tilde{\mathbf{F}} = \mathbf{F}^{-1} \quad (7)$$

which states the dual basis vectors are the rows of the inverse of the matrix whose columns are the basis vectors (and vice versa). When the vector set is a basis, \mathbf{F} is necessarily square and from (3) and (4), one can show

$$\mathbf{F}\tilde{\mathbf{F}}=\tilde{\mathbf{F}}\mathbf{F}. \quad (8)$$

Because this system requires two basis sets, the expansion basis and the dual basis, it is called biorthogonal.

If the basis vectors are not only independent but orthonormal, the basis set is its own dual and the inverse of \mathbf{F} is simply its transpose.

$$\tilde{\mathbf{F}} = \mathbf{F}^T \quad (9)$$

When done in Hilbert spaces, this decomposition is sometimes called an abstract Fourier expansion.

2 Frames and Tight Frames

If a set of vectors spans a space but are not linearly independent, (1) still holds but it is no longer unique. The set of vectors is called a frame for the space [6][2][5][3] and are redundant in the sense there are more than necessary for a basis. The finite dimensional matrix version of this case would have \mathbf{F} in (3) with more columns than rows but with full row rank. The dual frame vectors are also not unique but a set can be found such that (4) and, therefore, (5) holds (but (8) does not). A set of dual frame vectors could be found by adding a set of arbitrary but independent rows to \mathbf{F} until it is square, inverting it, then taking the first N columns to form $\tilde{\mathbf{F}}$ whose rows will be a set of dual frame vectors. This method of construction shows the non-uniqueness of the dual frame vectors. This non-uniqueness is often resolved by minimizing some other parameter of the system [3].

If the matrix operations are implementing a frame decomposition and the rows of \mathbf{F} are orthonormal, $\tilde{\mathbf{F}}=\mathbf{F}^T$ and the vector set is called a tight frame [6][3]. If the frame vectors are normalized to $\|\mathbf{x}_k\| = 1$, the decomposition in (1) becomes

$$\mathbf{x} = \frac{1}{A} \sum_n \langle \mathbf{x}, \tilde{\mathbf{x}}_n \rangle \mathbf{x}_n \quad (10)$$

where the constant A is a measure of the redundancy of the expansion which has more expansion vectors than necessary [3].

The matrix form is

$$\mathbf{x} = \frac{1}{A} \mathbf{F}\mathbf{F}^T \mathbf{x} \quad (11)$$

where \mathbf{F} has more columns than rows. Examples can be found in [1].

Frames and tight frames don't seem to be particularly useful in finite dimensions, but become important in infinite dimensional signal analysis, especially using the new idea of wavelet basis functions [3].

In an infinite dimensional vector space, if basis vectors are chosen such that all expansion converge very rapidly, the basis is called an unconditional basis and is near optimal for a wide class of signal representation and processing problems. This is discussed by Donoho in .

Still another view of a matrix operator being a change of basis can be developed using the eigenvectors (or singular values) of an operator as the basis vectors. Then a signal can be decomposed into its eigenvector components which are then simply multiplied by the scalar eigenvalues to accomplish the same task as a general matrix multiplication. This is an interesting idea but will not be developed here.

3 Change of Signal

If both \mathbf{x} and \mathbf{b} are considered to be signals in the same coordinate or basis system, the matrix operator \mathbf{A} is generally square. It may or may not be of full rank and it may or may not have a variety of other properties, but both \mathbf{x} and \mathbf{b} are viewed in the same coordinate system.

One method of understanding and generating matrices of this sort is to construct them as a product of first a decomposition operator, then a modification operator in the new basis system, followed by a recomposition operator. For example, one could first multiply a signal by the DFT operator which will change it into the frequency domain. One (or more) of the frequency coefficients could be removed (set to zero) and the remainder multiplied by the inverse DFT operator to give a signal back in the time domain but changed by having a frequency component removed. That is a form of signal filtering.

It would be instructive for the reader to make sense out of the cryptic statement "the DFT diagonalizes the cyclic convolution matrix" to add to the ideas in this note.

References

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