BESOV SPACES AND NONLINEAR APPROXIMATION[*]

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Abstract

The following is a short introduction to Besov spaces and their characterization by means of approximation procedures as well as wavelet decompositions.

A natural idea for approximating a function f by wavelets is to retain in the N largest contributions in the norm in which we plan to measure the error. In the case where this norm is L^p, this is given by

$$A_N f := \sum_{\lambda \in E_{N,p}(f)} d_\lambda \psi_\lambda, \qquad (1)$$

where $E_{N,p}(f)$ is the set of indices of the N largest $\| d_\lambda \psi_\lambda \|_{L^p}$. This set depends on the function f, making this approximation process **nonlinear**. Other instances of nonlinear approximation are discussed in [4].

An important result established in [5] states that $\| f - A_N f \|_{L^p} \sim N^{-r/d}$ is achieved for functions $f \in B_{q,q}^r$ where $1/q = 1/p + r/d$. Note that this relation between p and q corresponds to a critical case of the Sobolev embedding of $B_{q,q}^r$ into L^p. In particular, $B_{q,q}^r$ is not contained in $B_{p,p}^\epsilon$ for any $\epsilon > 0$, so that **no decay rate can be achieved by a linear approximation process** for all the functions f in the space $B_{q,q}^r$. (For **some** functions in $B_{q,q}^r$, which happen to also lie in spaces for which an independent linear approximation theorem can be written, it is of course possible to get a linear approximation rate; the point here is that this is possible only via such additional information.)

Note also that for large values of r, the parameter q given by $1/q = 1/p + r/d$ is smaller than 1. In such a situation the space $B_{q,q}^s$ is not a Banach space any more and is only a quasi-norm (it fails to satisfy the triangle inequality $\| x + y \| \leq \| x \| + \| y \|$). However, this space is still contained in L^1 (by a Sobolev-type embedding) and its characterization by means of wavelets coefficients according to still holds. Letting q go to zero as r goes to infinity allows the presence of singularities in the functions of $B_{q,q}^r$ even when r is large: for example, a function which is piecewise $\mathcal{C}^n$ on an interval except at a finite number of isolated points of discontinuities belongs to all $B_{q,q}^r$ for $q < 1/s$ and $r < n$. This is a particular instance where a non-linear approximation process will perform substantially better than a linear projection.

References

[1] R. Adams. *Sobolev Spaces*. Academic Press, 1975.

[2] A. Cohen. *Numerical Analysis of Wavelet Methods*. Elsevier, 2003.

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[3] Ingrid Daubechies. *Ten Lectures on Wavelets*. SIAM, Philadelphia, PA, 1992. Notes from the 1990 CBMS-NSF Conference on Wavelets and Applications at Lowell, MA.

[4] R. DeVore. *Nonlinear Approximation*. Acta Numerica, 1998.

[5] B. Jawerth R. DeVore and V. Popov. Compression of wavelet decompositions. *American Journal of Math*, 114:737–285, 1992.

[6] H. Triebel. *Theory of Function Spaces*. Birkhauser, 1983.