

# EMBEDDINGS\*

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## Abstract

The following is a short introduction to Besov spaces and their characterization by means of approximation procedures as well as wavelet decompositions.

Sobolev, Besov and Bessel-potential spaces satisfy two obvious embedding relations:

- For fixed  $p$  (and arbitrary  $q$  in the case of Besov spaces), the spaces get larger as  $s$  decreases.
- In the case where  $\Omega$  a bounded domain, for fixed  $s$  (and fixed  $q$  in the case of Besov spaces), the spaces get larger as  $p$  decrease, since  $\|f\|_{L^{p_1}} \leq C\|f\|_{L^{p_2}}$  if  $p_1 \leq p_2$ .

A less trivial type of embedding is known as **Sobolev embedding**. In the case of Sobolev spaces, it states that the continuous embedding

$$W^{s_1, p_1} \subset W^{s_2, p_2} \text{ if } p_1 \leq p_2 \text{ and } s_1 - s_2 \geq d(1/p_1 - 1/p_2), \quad (1)$$

holds except in the case where  $p_2 = +\infty$  and  $s_2$  is an integer, for which one needs to assume  $s_1 - s_2 > d(1/p_1 - 1/p_2)$ . For example in the univariate case, any  $H^1$  function has also  $C^{1/2}$  smoothness. In the case of Besov spaces the embedding relation are given by

$$B_{p_1, p_1}^{s_1} \subset B_{p_2, p_2}^{s_2} \text{ if } p_1 \leq p_2 \text{ and } s_1 - s_2 \geq d(1/p_1 - 1/p_2), \quad (2)$$

with no other restrictions on the indices  $s_1, s_2 \geq 0$ . In the case where  $\Omega$  is a bounded domain, these embedding are compact if and only if the strict inequality  $s_1 - s_2 > d(1/p_1 - 1/p_2)$  holds. The proof of these embeddings can be found in [1] for Sobolev spaces and [6] for Besov spaces.

As an exercise, let us see how these embeddings can be used to derive the range of  $r$  such that  $B_{2,q}^r([0, 1])$  can contain discontinuous functions. If  $r > 1/2$ , then there exists  $\epsilon > 0$  such that  $r - 2\epsilon > 1/2$ ; We remark that  $B_{2,q}^r \subset B_{2,q}^{r-\epsilon} \subset B_{\infty, \infty}^\epsilon = C^\epsilon$ , so all functions in  $B_{2,q}^r$  are continuous. Therefore only  $B_{2,q}^r$  with  $r \leq 1/2$  can contain discontinuous functions. In the limiting case  $r = 1/2$ , a closer inspection reveals that the functions in  $B_{2,q}^{1/2}$  are continuous if  $q < \infty$ , while  $B_{2, \infty}^{1/2}$  includes discontinuous functions, such as the characteristic function of an interval  $[0, a]$  for  $0 < a < 1$ .

## References

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