

LINEAR TIME INVARIANT SYSTEMS*

Thanos Antoulas
JP Slavinsky

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Abstract

Discussion of linear time invariant systems.

1 Introduction

Linearity and time invariance are two system properties that greatly simplify the study of systems that exhibit them. In our study of signals and systems, we will be especially interested in systems that demonstrate both of these properties, which together allow the use of some of the most powerful tools of signal processing.

2 Linear Time Invariant Systems

2.1 Linear Systems

If a system is linear, this means that when an input to a given system is scaled by a value, the output of the system is scaled by the same amount.

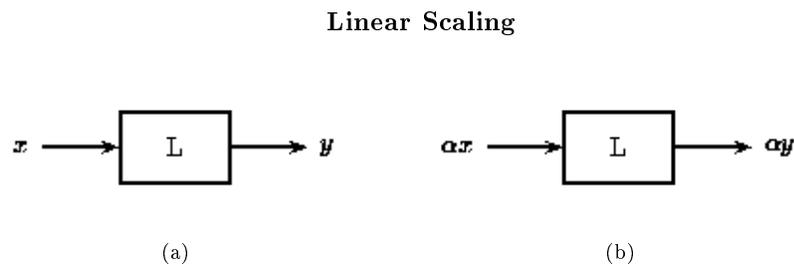


Figure 1

In Figure 1(a) above, an input x to the linear system L gives the output y . If x is scaled by a value α and passed through this same system, as in Figure 1(b), the output will also be scaled by α .

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A linear system also obeys the principle of superposition. This means that if two inputs are added together and passed through a linear system, the output will be the sum of the individual inputs' outputs.

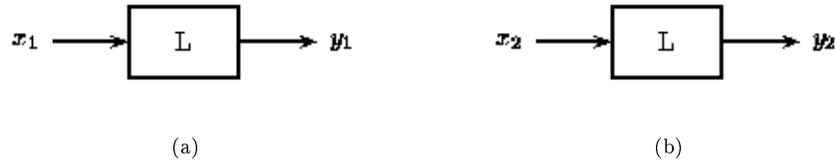


Figure 2

Superposition Principle

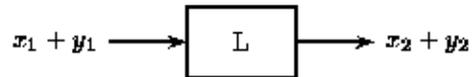


Figure 3: If Figure 2 is true, then the principle of superposition says that Figure 3 (Superposition Principle) is true as well. This holds for linear systems.

That is, if Figure 2 is true, then Figure 3 (Superposition Principle) is also true for a linear system. The scaling property mentioned above still holds in conjunction with the superposition principle. Therefore, if the inputs x and y are scaled by factors α and β , respectively, then the sum of these scaled inputs will give the sum of the individual scaled outputs:

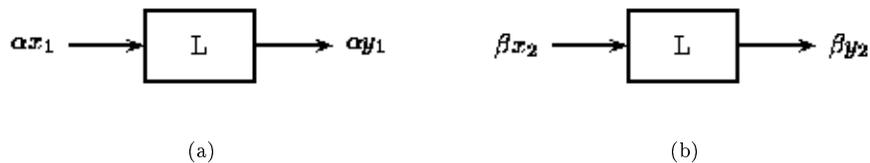


Figure 4

Superposition Principle with Linear Scaling

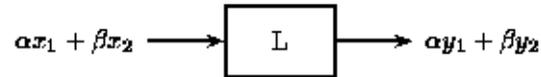


Figure 5: Given Figure 4 for a linear system, Figure 5 (Superposition Principle with Linear Scaling) holds as well.

Example 1

Consider the system H_1 in which

$$H_1(f(t)) = tf(t) \quad (1)$$

for all signals f . Given any two signals f, g and scalars a, b

$$H_1(af(t) + bg(t)) = t(af(t) + bg(t)) = atf(t) + btg(t) = aH_1(f(t)) + bH_1(g(t)) \quad (2)$$

for all real t . Thus, H_1 is a linear system.

Example 2

Consider the system H_2 in which

$$H_2(f(t)) = (f(t))^2 \quad (3)$$

for all signals f . Because

$$H_2(2t) = 4t^2 \neq 2t^2 = 2H_2(t) \quad (4)$$

for nonzero t , H_2 is not a linear system.

2.2 Time Invariant Systems

A time-invariant system has the property that a certain input will always give the same output (up to timing), without regard to when the input was applied to the system.

Time-Invariant Systems

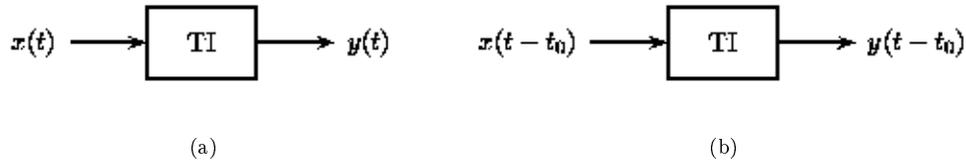


Figure 6: Figure 6(a) shows an input at time t while Figure 6(b) shows the same input t_0 seconds later. In a time-invariant system both outputs would be identical except that the one in Figure 6(b) would be delayed by t_0 .

In this figure, $x(t)$ and $x(t - t_0)$ are passed through the system TI. Because the system TI is time-invariant, the inputs $x(t)$ and $x(t - t_0)$ produce the same output. The only difference is that the output due to $x(t - t_0)$ is shifted by a time t_0 .

Whether a system is time-invariant or time-varying can be seen in the differential equation (or difference equation) describing it. **Time-invariant systems are modeled with constant coefficient equations.** A constant coefficient differential (or difference) equation means that the parameters of the system are **not** changing over time and an input now will give the same result as the same input later.

Example 3

Consider the system H_1 in which

$$H_1(f(t)) = tf(t) \quad (5)$$

for all signals f . Because

$$S_T(H_1(f(t))) = S_T(tf(t)) = (t - T)f(t - T) \neq tf(t - T) = H_1(f(t - T)) = H_1(S_T(f(t))) \quad (6)$$

for nonzero T , H_1 is not a time invariant system.

Example 4

Consider the system H_2 in which

$$H_2(f(t)) = (f(t))^2 \quad (7)$$

for all signals f . For all real T and signals f ,

$$S_T(H_2(f(t))) = S_T(f(t)^2) = (f(t - T))^2 = H_2(f(t - T)) = H_2(S_T(f(t))) \quad (8)$$

for all real t . Thus, H_2 is a time invariant system.

2.3 Linear Time Invariant Systems

Certain systems are both linear and time-invariant, and are thus referred to as LTI systems.

Linear Time-Invariant Systems

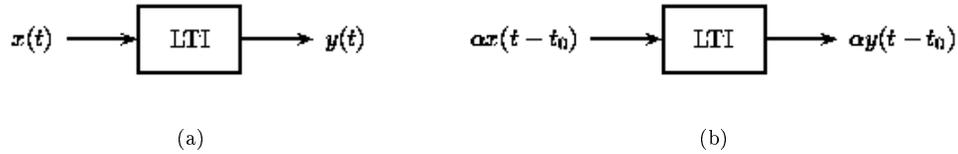


Figure 7: This is a combination of the two cases above. Since the input to Figure 7(b) is a scaled, time-shifted version of the input in Figure 7(a), so is the output.

As LTI systems are a subset of linear systems, they obey the principle of superposition. In the figure below, we see the effect of applying time-invariance to the superposition definition in the linear systems section above.

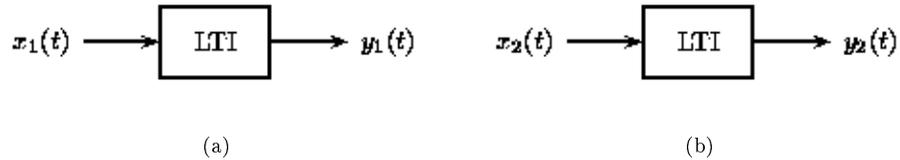


Figure 8

Superposition in Linear Time-Invariant Systems



Figure 9: The principle of superposition applied to LTI systems

2.3.1 LTI Systems in Series

If two or more LTI systems are in series with each other, their order can be interchanged without affecting the overall output of the system. Systems in series are also called cascaded systems.

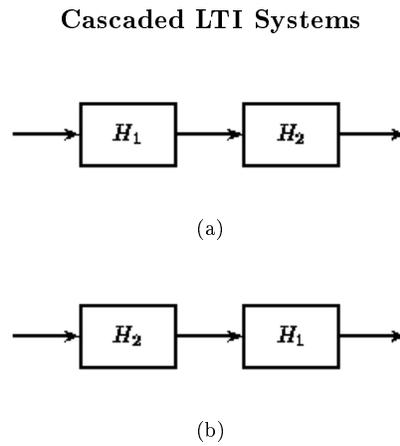


Figure 10: The order of cascaded LTI systems can be interchanged without changing the overall effect.

2.3.2 LTI Systems in Parallel

If two or more LTI systems are in parallel with one another, an equivalent system is one that is defined as the sum of these individual systems.

Parallel LTI Systems

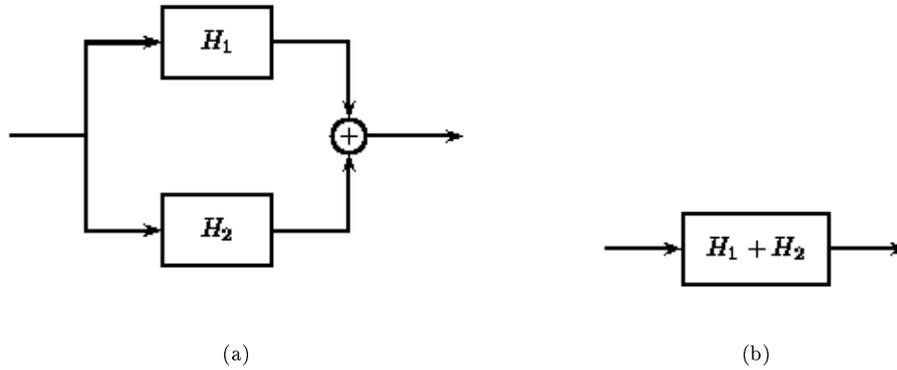


Figure 11: Parallel systems can be condensed into the sum of systems.

Example 5

Consider the system H_3 in which

$$H_3(f(t)) = 2f(t) \quad (9)$$

for all signals f . Given any two signals f, g and scalars a, b

$$H_3(af(t) + bg(t)) = 2(af(t) + bg(t)) = a2f(t) + b2g(t) = aH_3(f(t)) + bH_3(g(t)) \quad (10)$$

for all real t . Thus, H_3 is a linear system. For all real T and signals f ,

$$S_T(H_3(f(t))) = S_T(2f(t)) = 2f(t - T) = H_3(f(t - T)) = H_3(S_T(f(t))) \quad (11)$$

for all real t . Thus, H_3 is a time invariant system. Therefore, H_3 is a linear time invariant system.

Example 6

As has been previously shown, each of the following systems are not linear or not time invariant.

$$H_1(f(t)) = tf(t) \quad (12)$$

$$H_2(f(t)) = (f(t))^2 \quad (13)$$

Thus, they are not linear time invariant systems.

3 Linear Time Invariant Demonstration

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Figure 12: Interact(when online) with the Mathematica CDF above demonstrating Linear Time Invariant systems. To download, right click and save file as .cdf.

4 LTI Systems Summary

Two very important and useful properties of systems have just been described in detail. The first of these, linearity, allows us the knowledge that a sum of input signals produces an output signal that is the summed original output signals and that a scaled input signal produces an output signal scaled from the original output signal. The second of these, time invariance, ensures that time shifts commute with application of the system. In other words, the output signal for a time shifted input is the same as the output signal for the original input signal, except for an identical shift in time. Systems that demonstrate both linearity and time invariance, which are given the acronym LTI systems, are particularly simple to study as these properties allow us to leverage some of the most powerful tools in signal processing.