Defining the State for an n-th Order Differential Equation*

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Abstract

Defining the State for an n-th Order Differential Equation

Consider the n-th order linear differential equation:

$$q(s)\,y(t) = p(s)\,u(t) \tag{1}$$

where $s = \frac{d}{d(t)}$ and where

$$q(s) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0 \tag{2}$$

$$p(s) = \beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0 \tag{3}$$

One way to define state variables is by introducing the auxiliary variable w which satisfies the differential equation:

$$q(s)\,w(t) = u(t) \tag{4}$$

The state variables can then be chosen as derivatives of w . Furthermore the output is related to this auxiliary variable as follows:

$$y(t) = p(s)\,w(t) \tag{5}$$

The proof in the next three equations shows that the introduction of this variable w does not change the system in any way. The first equation uses a simple substition based on the differential equation (4). Then the order of $p(s)$ and $q(s)$ are interchanged. Lastly, y is substituted in place of $p(s)\,w(t)$ (using output equation (5)). The result is the original equation describing our system.

$$p(s)\,q(s)\,w(t) = p(s)\,u(t) \tag{6}$$

$$q(s)\,p(s)\,w(t) = p(s)\,u(t) \tag{7}$$

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$$q(s)\,y(t) = p(s)\,u(t) \tag{8}$$

Using this auxillary variable, we can directly write the A, B and C matrices. A is the companion-form matrix; its last row (except for a 0 in the first position) contains the alpha coefficients from the $q(s)$:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \ddots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 & \dots & -\alpha_{n-1} \end{pmatrix} \tag{9}$$

The B vector has zeros except for the n-th row which is a 1.

$$B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \tag{10}$$

C can be expressed as

$$C = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \end{pmatrix} \tag{11}$$

When all of these conditions are met, the state is

$$x = \begin{pmatrix} w \\ sw \\ s^2 w \\ \vdots \\ s^{n-1} w \end{pmatrix} \tag{12}$$

In conclusion, if the degree of p is less than that of q, we can obtain a state-space representation by inserting the coefficcients of p and q in the matrices A, B and C as shown above.