Connexions module: m2111

PARTIAL FRACTION EXPANSION*

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Abstract

This module describes the method of partial fraction expansion, in which a ratio of polynomials can be split into a sum of small polynomials. The Heaviside cover-up method is discussed in detail with examples. Finding a partial fraction expansion in matlab is also discussed.

Splitting up a ratio of large polynomials into a sum of ratios of small polynomials can be a useful tool, especially for many problems involving Laplace-like transforms. This technique is known as partial fraction expansion. Here's an example of one ratio being split into a sum of three simpler ratios:

$$\frac{8x^2 + 3x - 21}{x^3 - 7x - 6} = \frac{1}{x + 2} + \frac{3}{x - 3} + \frac{4}{x + 1} \tag{1}$$

There are several methods for expanding a rational function via partial fractions. These include the method of clearing fractions, the Heaviside "cover-up" method, and different combinations of these two. For many cases, the Heaviside "cover-up" method is the easiest, and is therefore the method that we will introduce here. For a more complete discussion, see Signal Processing and Linear Systems by B.P. Lathi, Berkeley-Cambridge Press, 1998, pp-24-33. Some of the material below is based upon this book.

1 Heaviside "Cover-Up" Method

1.1 No Repeated Roots

Let's say we have a proper function $G(x) = \frac{N(x)}{D(x)}$ (by proper we mean that the degree m of the numerator N(x) is less than the degree p of denominator D(x)). In this section we assume that there are no repeated roots of the polynomial D(x).

The first step is to factor the denominator D(x):

$$G(x) = \frac{N(x)}{(x - a_1)(x - a_2)\dots(x - a_p)}$$
(2)

where $a_1 \ldots a_p$ are the roots of D(x). We can then rewrite G(x) as a sum of partial fractions:

$$G(x) = \frac{\alpha_1}{x - a_1} + \frac{\alpha_2}{x - a_2} + \dots + \frac{\alpha_p}{x - a_p}$$
(3)

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where $a_1
ldots a_p$ are constants. Now, to complete the process, we must determine the values of these α coefficients. Let's look at how to find α_1 . If we multiply both sides of the equation of G(x) as a sum of partial fractions (3) by $x - a_1$ and then let $x = a_1$, all of the terms on the right-hand side will go to zero except for α_1 . Therefore, we'll be left over with:

$$\alpha_1 = (x - a_1) G(x)|_{x=a_1}$$
 (4)

We can easily generalize this to a solution for any one of the unknown coefficients:

$$\alpha_r = (x - a_r) G(x) \big|_{x = a_r} \tag{5}$$

This method is called the "cover-up" method because multiplying both sides by $x - a_r$ can be thought of as simply using one's finger to cover up this term in the denominator of G(x). With a finger over the term that would be canceled by the multiplication, you can plug in the value $x = a_r$ and find the solution for α_r .

Example 1

In this example, we'll work through the partial fraction expansion of the ratio of polynomials introduced above. Before doing a partial fraction expansion, you must make sure that the ratio you are expanding is proper. If it is not, you should do long division to turn it into the sum of a proper fraction and a polynomial. Once this is done, the first step is to factor the denominator of the function:

$$\frac{8x^2 + 3x - 21}{x^3 - 7x - 6} = \frac{8x^2 + 3x - 21}{(x+2)(x-3)(x+1)} \tag{6}$$

Now, we set this factored function equal to a sum of smaller fractions, each of which has one of the factored terms for a denominator.

$$\frac{8x^2 + 3x - 21}{(x+2)(x-3)(x+1)} = \frac{\alpha_1}{x+2} + \frac{\alpha_2}{x-3} + \frac{\alpha_3}{x+1}$$
 (7)

To find the alpha terms, we just cover up the corresponding denominator terms in G(x) and plug in the root associated with the alpha:

$$\alpha_{1} = (x+2) G(x)|_{x=-2}$$

$$= \frac{8x^{2}+3x-21}{(x-3)(x+1)}|_{x=-2}$$

$$= 1$$
(8)

$$\alpha_{2} = (x-3) G(x)|_{x=3}$$

$$= \frac{8x^{2}+3x-21}{(x+2)(x+1)}|_{x=3}$$

$$= 3$$
(9)

$$\alpha_{3} = (x+3) G(x)|_{x=-1}$$

$$= \frac{8x^{2}+3x-21}{(x+2)(x-3)}|_{x=-1}$$

$$= 4$$
(10)

We now have our completed partial fraction expansion:

$$\frac{8x^2 + 3x - 21}{(x+2)(x-3)(x+1)} = \frac{1}{x+2} + \frac{3}{x-3} + \frac{4}{x+1}$$
(11)

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1.2 Repeated Roots

When the function G(x) has a repeated root in its denominator, as in

$$G(x) = \frac{N(x)}{(x-b)^{r}(x-a_{1})(x-a_{2})\dots(x-a_{j})}$$
(12)

Somewhat more special care must be taken to find the partial fraction expansion. The non-repeated terms are expanded as before, but for the repeated root, an extra fraction is added for each instance of the repeated root:

$$G(x) = \frac{\beta_0}{(x-b)^r} + \frac{\beta_1}{(x-b)^{r-1}} + \dots + \frac{\beta_{r-1}}{x-b} + \frac{\alpha_1}{x-a_1} + \frac{\alpha_2}{x-a_2} + \dots + \frac{\alpha_j}{x-a_j}$$
(13)

All of the alpha constants can be found using the non-repeated roots method above. Finding the beta coefficients (which are due to the repeated root) has the same Heaviside feel to it, except that this time we will add a twist by using the derivative to eliminate some unwanted terms.

Starting off directly with the cover-up method, we can find β_0 . By multiplying both sides by $(x-b)^r$, we'll get:

$$(x-b)^{r}G(x) = \beta_{0} + \beta_{1}(x-b) + \dots + \beta_{r-1}(x-b)^{r-1} + \alpha_{1}\frac{(x-b)^{r}}{x-a_{1}} + \alpha_{2}\frac{(x-b)^{r}}{x-a_{2}} + \dots + \alpha_{j}\frac{(x-b)^{r}}{x-a_{j}}$$
(14)

Now that we have "covered up" the $(x-b)^r$ term in the denominator of G(x), we plug in x=b to each side; this cancels every term on the right-hand side except for β_0 , leaving the formula

$$\beta_0 = (x - b)^r G(x)|_{x=b} \tag{15}$$

To find the other values of the beta coefficients, we can take advantage of the derivative. By taking the derivative of the equation after cover-up (14) (with respect to x the right-hand side becomes β_1 plus terms containing an x-b in the numerator. Again, plugging in x=b eliminates everything on the right-hand side except for β_1 , leaving us with a formula for β_1 :

$$\beta_1 = \frac{d(x-b)^r G(x)}{dx} \Big|_{x=b} \tag{16}$$

Generalizing over this pattern, we can continue to take derivatives to find the other beta terms. The solution for all beta terms is

$$\beta_k = \frac{1}{k!} \frac{d^k (x-b)^r G(x)}{dx^k} \Big|_{x=b}$$
 (17)

NOTE: To check if you've done the partial fraction expansion correctly, just add all of the partial fractions together to see if their sum equals the original ratio of polynomials.

2 Finding Partial Fractions in Matlab

Matlab can be a useful tool in finding partial fraction expansions when the ratios become too unwieldy to expand by hand. It can handle symbolic variables. For example, if you type syms s, s will be treated as a symbolic variable. You can then use it as such when you make function assignments.

If you've done this and have then made a function, say H(s), which is a ratio of two polynomials in the symbolic variable s, there are two ways to get the partial fraction expansion of it. A trick way is to say diff(int(H)). When you use these functions together, Matlab gives back H expanded into partial fractions. There's also a more formal way to do it using the residue command. Type help residue in Matlab for details.