Diagonalizability*

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Abstract

Diagonalizability of Matrices

A diagonal matrix is one whose elements not on the diagonal are equal to 0. The following matrix is one example.

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

A matrix A is diagonalizable if there exists a matrix $V \in \mathbb{R}^{n \times n}$, $\det V \neq 0$ such that $VAV^{-1} = \Lambda$ is diagonal. In such a case, the diagonal entries of Λ are the eigenvalues of A.

Let's take an eigenvalue decomposition example to work backwards to this result.

Assume that the matrix A has eigenvectors v and w and the respective eigenvalues λ_v and λ_w:

$$Av = \lambda_v v$$

$$Aw = \lambda_w w$$

We can combine these two equations into an equation of matrices:

$$A \begin{pmatrix} v & w \end{pmatrix} = \begin{pmatrix} v & w \end{pmatrix} \begin{pmatrix} \lambda_v & 0 \\ 0 & \lambda_v \end{pmatrix}$$

To simplify this equation, we can replace the eigenvector matrix with V and the eigenvalue matrix with Λ.

$$AV = V\Lambda$$

Now, by multiplying both sides of the equation by V^{-1}, we see the diagonalizability equation discussed above.

$$A = V\Lambda V^{-1} \quad (1)$$

*Version 2.13: Nov 12, 2003 10:09 pm -0600
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When is such a diagonalization possible? The condition is that the algebraic multiplicity equal the geometric multiplicity for each eigenvalue, $\alpha_i = \gamma_i$. This makes sense; basically, we are saying that there are as many eigenvectors as there are eigenvalues. If it were not like this, then the V matrices would not be square, and therefore could not be inverted as is required by the diagonalizability equation (1). Remember that the eigenspace associated with a certain eigenvalue λ is given by $\ker(A - \lambda I)$.

This concept of diagonalizability will come in handy in different linear algebra manipulations later. We can however, see a time-saving application of it now. If the matrix A is diagonalizable, and we know its eigenvalues λ_i, then we can immediately find the eigenvalues of A^2:

$$A^2 = \left(V \Lambda V^{-1}\right)\left(V \Lambda V^{-1}\right) = V \Lambda^2 V^{-1}$$

The eigenvalues of A^2 are simply the eigenvalues of A, squared.