Complex Numbers: An Electric Field Computation*

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We have established that vectors may be used to code complex numbers. Conversely, complex numbers may be used to code or represent the orthogonal components of any two-dimensional vector. This makes them invaluable in electromagnetic field theory, where they are used to represent the components of electric and magnetic fields.

The basic problem in electromagnetic field theory is to determine the electric or magnetic field that is generated by a static or dynamic distribution of charge. The key idea is to isolate an infinitesimal charge, determine the field set up by this charge, and then to sum the fields contributed by all such infinitesimal charges. This idea is illustrated in Figure 1, where the charge λ, uniformly distributed over a line segment of length dx at point $-x$, produces a field $dE\,(x)$ at the test point $(0,h)$. The field $dE\,(x)$ is a "vector" field (as opposed to a "scalar" field), with components $E_1\,(x)$ and $E_2\,(x)$. The intensity or field strength of the field $dE\,(x)$ is

$$|dE\,(x)\,| = \frac{\lambda dx}{4\pi\varepsilon_0\,(h^2+x^2)} \tag{1}$$

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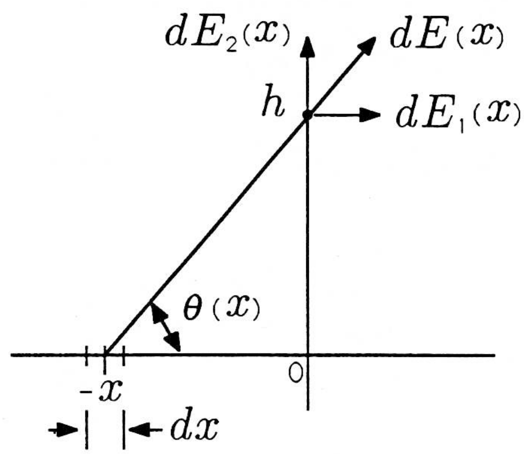

Figure 1: Infinitesimal Charge λdx Producing Field $dE(x)$

But the field strength is directed at angle $\theta(x)$, as illustrated in Figure 1. The field $dE(x)$ is real with components $dE_1(x)$ and $dE_2(x)$, but we code it as a complex field. We say that the "complex" field at test point $(0, h)$ is

$$dE(x) = \frac{\lambda dx}{4\pi\varepsilon_0 (h^2 + x^2)} e^{j\theta(x)} \tag{2}$$

with components $dE_1(x)$ and $dE_2(x)$. That is,

$$dE(x) = dE_1(x) + jdE_2(x) \tag{3}$$

$$dE_1(x) = \frac{\lambda dx}{4\pi\varepsilon_0 (h^2 + x^2)} \ cos\ \theta(x) \tag{4}$$

$$dE_2(x) = \frac{\lambda dx}{4\pi\varepsilon_0 (h^2 + x^2)} \ sin\ \theta(x). \tag{5}$$

For charge uniformally distributed with density λ along the x-axis, the total field at the test point $(0, h)$ is obtained by integrating dE:

$$\int_{-\infty}^{\infty} dE(x) = \int_{-\infty}^{\infty} \frac{\lambda}{4\pi\varepsilon_0 (h^2 + x^2)} [cos\theta(x) + jsin\theta(x)]\, dx. \tag{6}$$

The functions cos $\theta(x)$ and $sin\theta(x)$ are

$$cos\theta(x) = \frac{x}{(x^2 + h^2)^{1/2}}; \quad sin\theta(x) = \frac{h}{(x^2 + h^2)^{1/2}} \tag{7}$$

We leave it as a problem to show that the real component E_1 of the field is zero. The imaginary component E_2 is

$$E = jE_2 = j\int_{-\infty}^{\infty} \frac{\lambda h}{4\pi\varepsilon_0} \frac{dx}{(x^2 + h^2)^{3/2}} \tag{8}$$

$$= j\frac{\lambda h}{4\pi\varepsilon_0}\frac{x}{h^2(x^2+h^2)^{1/2}}|_{-\infty}^{\infty} \tag{9}$$

$$= j\frac{\lambda h}{4\pi\varepsilon_0}\left[\frac{1}{h^2}+\frac{1}{h^2}\right] = j\frac{\lambda}{2\pi\varepsilon_0 h} \tag{10}$$

$$E_2 = \frac{\lambda}{2\pi\varepsilon_0 h}. \tag{11}$$

We emphasize that the field at $(0, h)$ is a *real* field. Our imaginary answer simply says that the real field is *oriented* in the vertical direction because we have used the imaginary part of the complex field to code the vertical component of the real field.

Exercise 1

Show that the horizontal component of the field E is zero. Interpret this finding physically.

From the symmetry of this problem, we conclude that the field around the infinitely long wire of Figure 1 is radially symmetric. So, in polar coordinates, we could say

$$E\left(r,\theta\right) = \frac{\lambda}{2\pi\varepsilon_0 r} \tag{12}$$

which is independent of θ. If we integrated the field along a radial line perpendicular to the wire, we would measure the voltage difference

$$V\left(r_1\right) - V\left(r_0\right) = \int_{r_0}^{r_1}\frac{\lambda}{2\pi\varepsilon_0 r}dr = \frac{\lambda}{2\pi\varepsilon_0}\left[logr_1 - logr_0\right]. \tag{13}$$

An electric field has units of volts/meter, a charge density λ has units of coulombs/meter, and ε_0 has units of coulombs/volt-meter; voltage has units of volts (of course).