Phasors: Beating between Tones*

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NOTE: This module is part of the collection, *A First Course in Electrical and Computer Engineering*. The LaTeX source files for this collection were created using an optical character recognition technology, and because of this process there may be more errors than usual. Please contact us if you discover any errors.

Perhaps you have heard two slightly mistuned musical instruments play pure tones whose frequencies are close but not equal. If so, you have sensed a beating phenomenon wherein a pure tone seems to wax and wane. This waxing and waning tone is, in fact, a tone whose frequency is the average of the two mismatched frequencies, amplitude modulated by a tone whose "beat" frequency is half the difference between the two mismatched frequencies. The effect is illustrated in Figure 1. Let's see if we can derive a mathematical model for the beating of tones.

We begin with two pure tones whose frequencies are $\omega_0+\nu$ and $\omega_0-\nu$ (for example, $\omega_0 = 2\pi\times 10^3 rad/sec$ and $\nu = 2\pi rad/sec$). The average frequency is ω_0, and the difference frequency is 2ν. What you hear is the sum of the two tones:

$$x(t) = A_1\, cos\left[(\omega_0+\nu)\,t+\phi_1\right] + A_2 cos\left[(\omega_0-\nu)\,t+\phi_2\right]. \tag{1}$$

The first tone has amplitude A_1 and phase ϕ_1 ; the second has amplitude A_2 and phase ϕ_2. We will assume that the two amplitudes are equal to A. Furthermore, whatever the phases, we may write them as

$$\begin{array}{ccc} \phi_1 = \phi+\psi & \text{and} & \phi_2 = \phi-\psi \\ \phi = \frac{1}{2}(\phi_1+\phi_2) & \text{and} & \psi = \frac{1}{2}(\phi_1-\phi_2). \end{array} \tag{2}$$

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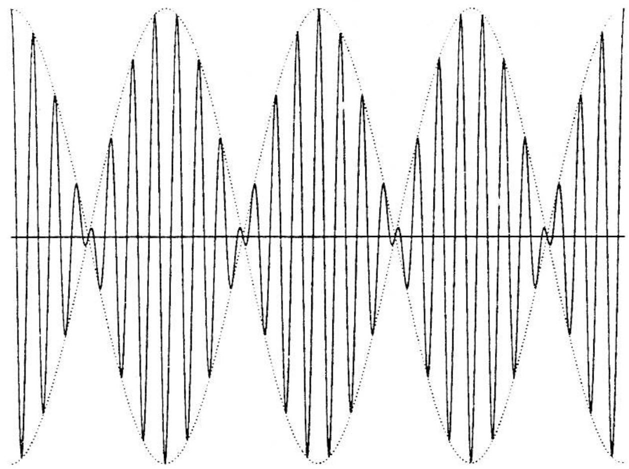

Figure 1: Beating between Tones

Recall our trick for representing $x(t)$ as a complex phasor:

$$\begin{array}{rcl} x(t) & = & A\,\mathrm{Re}\{e^{j[(\omega_0+\nu)t+\phi+\psi]},+,e^{j[(\omega_0-\nu)t+\phi-\psi]}\} \\ & = & A\,\mathrm{Re}\{e^{j(\omega_0 t+\phi)},\left[e^{j(\nu t+\psi)}+e^{-j(\nu t+\psi)}\right]\} \\ & = & 2A\,\mathrm{Re}\{e^{j(\omega_0 t+\phi)},cos,(\nu t+\psi)\} \\ & = & 2A\,cos\,(\omega_0 t+\phi)\,cos\,(\nu t+\psi)\,. \end{array} \quad (3)$$

This is an amplitude modulated wave, wherein a low frequency signal with beat frequency ν rad/sec modulates a high frequency signal with carrier frequency ω_0 rad/sec. Over short periods of time, the modulating term $cos\,(\nu t+\psi$ remains essentially constant while the carrier term $cos\,(\omega_0 t+\phi)$ turns out many cycles of its tone. For example, if t runs from 0 to $\frac{2\pi}{10\nu}$ (about 0.1 seconds in our example), then the modulating wave turns out just 1/10 cycle while the carrier turns out $\overline{1}0\overline{\nu}\omega_{\Delta}$ cycles (about 100 in our example). Every time νt changes by 2π radians, then the modulating term goes from a maximum (a wax) through a minimum (a wane) and back to a maximum. This cycle takes

$$\nu t=2\pi \Leftrightarrow t=\frac{2\pi}{\nu}\text{seconds}, \quad (4)$$

which is 1 second in our example. In this 1 second the carrier turns out 1000 cycles.

Exercise 1

Find out the frequency of A above middle C on a piano. Assume two pianos are mistuned by $\pm 1Hz\,(\pm 2\pi\text{rad/sec})$. Find their beat frequency ν and their carrier frequency ω_0.

Exercise 2

(MATLAB) Write a MATLAB program to compute and plot

$Acos\,[(\omega_0+\nu)\,t+\phi_1]\,,A\cos\,[(\omega_0-\nu)\,t+\phi_2]$, and their sum. Then compute and plot $2Acos\,(\omega_0 t+\phi)\,cos\,(\nu t+\psi)$. Verify that the sum equals this latter signal.