## PHASORS: BEATING BETWEEN TONES\*

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Perhaps you have heard two slightly mistuned musical instruments play pure tones whose frequencies are close but not equal. If so, you have sensed a beating phenomenon wherein a pure tone seems to wax and wane. This waxing and waning tone is, in fact, a tone whose frequency is the average of the two mismatched frequencies, amplitude modulated by a tone whose "beat" frequency is half the difference between the two mismatched frequencies. The effect is illustrated in Figure 1. Let's see if we can derive a mathematical model for the beating of tones.

We begin with two pure tones whose frequencies are  $\omega_0 + \nu$  and  $\omega_0 - \nu$  (for example,  $\omega_0 = 2\pi \times 10^3 rad/sec$ and  $\nu = 2\pi rad/sec$ ). The average frequency is  $\omega_0$ , and the difference frequency is  $2\nu$ . What you hear is the sum of the two tones:

$$x(t) = A_1 \cos \left[ (\omega_0 + \nu) t + \phi_1 \right] + A_2 \cos \left[ (\omega_0 - \nu) t + \phi_2 \right].$$
(1)

The first tone has amplitude  $A_1$  and phase  $\phi_1$ ; the second has amplitude  $A_2$  and phase  $\phi_2$ . We will assume that the two amplitudes are equal to A. Furthermore, whatever the phases, we may write them as

$$\phi_1 = \phi + \psi \quad \text{and} \quad \phi_2 = \phi - \psi$$
  

$$\phi = \frac{1}{2} (\phi_1 + \phi_2) \quad \text{and} \quad \psi = \frac{1}{2} (\phi_1 - \phi_2).$$
(2)

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Figure 1: Beating between Tones

Recall our trick for representing x(t) as a complex phasor:

$$x(t) = A \operatorname{Re} \{ e^{j((\omega_0 + \nu)t + \phi + \psi)}, +, e^{j((\omega_0 - \nu)t + \phi - \psi)} \}$$
  

$$= A \operatorname{Re} \{ e^{j((\omega_0 t + \phi))}, [e^{j(\nu t + \psi)} + e^{-j(\nu t + \psi)}] \}$$
  

$$= 2A \operatorname{Re} \{ e^{j((\omega_0 t + \phi))}, \cos((\nu t + \psi)) \}$$
  

$$= 2A \cos((\omega_0 t + \phi)) \cos((\nu t + \psi)).$$
(3)

This is an amplitude modulated wave, wherein a low frequency signal with beat frequency  $\nu$  rad/sec modulates a high frequency signal with carrier frequency  $\omega_0$  rad/sec. Over short periods of time, the modulating term  $\cos(\nu t + \psi)$  remains essentially constant while the carrier term  $\cos(\omega_0 t + \phi)$  turns out many cycles of its tone. For example, if t runs from 0 to  $\frac{2\pi}{10\nu}$  (about 0.1 seconds in our example), then the modulating wave turns out just 1/10 cycle while the carrier turns out  $\overline{10\nu\omega_{\Delta}}$  cycles (about 100 in our example). Every time  $\nu t$  changes by  $2\pi$  radians, then the modulating term goes from a maximum (a wax) through a minimum (a wane) and back to a maximum. This cycle takes

$$\nu t = 2\pi \Leftrightarrow t = \frac{2\pi}{\nu} \text{seconds},$$
(4)

which is 1 second in our example. In this 1 second the carrier turns out 1000 cycles.

## Exercise 1

Find out the frequency of A above middle C on a piano. Assume two pianos are mistuned by  $\pm 1Hz (\pm 2\pi \text{rad/sec})$ . Find their beat frequency  $\nu$  and their carrier frequency  $\omega_0$ .

## Exercise 2

(MATLAB) Write a MATLAB program to compute and plot

 $Acos[(\omega_0 + \nu)t + \phi_1], Acos[(\omega_0 - \nu)t + \phi_2], and their sum.$  Then compute and plot  $2Acos(\omega_0 t + \phi)cos(\nu t + \psi)$ . Verify that the sum equals this latter signal.