Basic Properties of Real Numbers: Rules of Exponents*

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Abstract

This module is from Elementary Algebra by Denny Burzynski and Wade Ellis, Jr. The symbols, notations, and properties of numbers that form the basis of algebra, as well as exponents and the rules of exponents, are introduced in this chapter. Each property of real numbers and the rules of exponents are expressed both symbolically and literally. Literal explanations are included because symbolic explanations alone may be difficult for a student to interpret. Objectives of this module: understand the product and quotient rules for exponents, understand the meaning of zero as an exponent.

1 Overview

- The Product Rule for Exponents
- The Quotient Rule for Exponents
- Zero as an Exponent

We will begin our study of the rules of exponents by recalling the definition of exponents.
Definition of Exponents
If x is any real number and n is a natural number, then

$$x^n = \underbrace{x \cdot x \cdot x \cdot ... \cdot x}_{n \text{ factors of } x}$$

An exponent records the number of identical factors in a multiplication.
Base Exponent Power
In x^n,

x is the **base**
n is the **exponent**
The number represented by x^n is called a **power**.

The term x^n is read as "x to the n th."

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2 The Product Rule for Exponents

The first rule we wish to develop is the rule for multiplying two exponential quantities having the **same base** and natural number exponents. The following examples suggest this rule:

Example 1

$$x^2 \cdot x^4 = \underbrace{xx}_{2} \cdot \underbrace{xxxx}_{4} = \underbrace{xxxxxx}_{6} = x^6$$

$$2 \text{ factors} + 4 \text{ factors} = 6 \text{ factors}$$

Example 2

$$a \cdot a^2 = \underbrace{a}_{1} \cdot \underbrace{aa}_{2} = \underbrace{aaa}_{3} = a^3$$

$$1 \text{ factors} + 2 \text{ factors} = 3$$

PRODUCT RULE FOR EXPONENTS
If x is a real number and n and m are natural numbers,

$$x^n x^m = x^{n+m}$$

To multiply two exponential quantities having the same base, add the exponents. Keep in mind that the exponential quantities being multiplied **must** have the **same base** for this rule to apply.

3 Sample Set A

Find the following products. All exponents are natural numbers.

Example 3
$x^3 \cdot x^5 = x^{3+5} = x^8$

Example 4
$a^6 \cdot a^{14} = a^{6+14} = a^{20}$

Example 5
$y^5 \cdot y = y^5 \cdot y^1 = y^{5+1} = y^6$

Example 6
$(x-2y)^8 (x-2y)^5 = (x-2y)^{8+5} = (x-2y)^{13}$

Example 7
$x^3 y^4 \neq (xy)^{3+4}$ Since the bases are not the same, the product rule does not apply.

4 Practice Set A

Find each product.

Exercise 1 *(Solution on p. 11.)*

$x^2 \cdot x^5$

Exercise 2 *(Solution on p. 11.)*

$x^9 \cdot x^4$

Exercise 3 *(Solution on p. 11.)*

$y^6 \cdot y^4$

Exercise 4 *(Solution on p. 11.)*

$c^{12} \cdot c^8$

Exercise 5 *(Solution on p. 11.)*

$(x+2)^3 \cdot (x+2)^5$

5 Sample Set B

We can use the first rule of exponents (and the others that we will develop) along with the properties of real numbers.

Example 8

$2x^3 \cdot 7x^5 = \boxed{2 \cdot 7 \cdot x^{3+5}} = 14x^8$

We used the commutative and associative properties of multiplication. In practice, we use these properties "mentally" (as signified by the drawing of the box). We don't actually write the second step.

Example 9

$4y^3 \cdot 6y^2 = \boxed{4 \cdot 6 \cdot y^{3+2}} = 24y^5$

Example 10

$9a^2b^6\left(8ab^4 2b^3\right) = \boxed{9 \cdot 8 \cdot 2a^{2+1}b^{6+4+3}} = 144a^3b^{13}$

Example 11

$5(a+6)^2 \cdot 3(a+6)^8 = \boxed{5 \cdot 3(a+6)^{2+8}} = 15(a+6)^{10}$

Example 12

$4x^3 \cdot 12 \cdot y^2 = 48x^3y^2$

Example 13

$$4a^{\triangle} \cdot 5a^{\star} = 20a^{\triangle + \star}$$

The bases are the same, so we add the exponents. Although we don't know exactly what number $\triangle + \star$ is, the notation $\triangle + \star$ indicates the addition.

6 Practice Set B

Perform each multiplication in one step.

Exercise 6 *(Solution on p. 11.)*

$3x^5 \cdot 2x^2$

Exercise 7 *(Solution on p. 11.)*

$6y^3 \cdot 3y^4$

Exercise 8 *(Solution on p. 11.)*

$4a^3b^2 \cdot 9a^2b$

Exercise 9 *(Solution on p. 11.)*

$x^4 \cdot 4y^2 \cdot 2x^2 \cdot 7y^6$

Exercise 10 *(Solution on p. 11.)*

$(x-y)^3 \cdot 4(x-y)^2$

Exercise 11 *(Solution on p. 11.)*

$8x^4y^2xx^3y^5$

Exercise 12 *(Solution on p. 11.)*

$2aaa^3\left(ab^2a^3\right)b6ab^2$

Exercise 13 *(Solution on p. 11.)*

$a^n \cdot a^m \cdot a^r$

7 The Quotient Rule for Exponents

The second rule we wish to develop is the rule for dividing two exponential quantities having the same base and natural number exponents.

The following examples suggest a rule for dividing two exponential quantities having the same base and natural number exponents.

Example 14

$\frac{x^5}{x^2} = \frac{xxxxx}{xx} = \frac{\overline{()xx})xxx}{\overline{()xx})} = xxx = x^3. \quad \text{Notice that } 5-2=3.$

Example 15

$\frac{a^8}{a^3} = \frac{aaaaaaaa}{aaa} = \frac{\overline{()aaa})aaaaa}{\overline{()aaa})} = aaaaa = a^5. \quad \text{Notice that } 8-3=5.$

QUOTIENT RULE FOR EXPONENTS

If x is a real number and n and m are natural numbers,

$$\frac{x^n}{x^m} = x^{n-m}, \quad x \neq 0 \ .$$

To divide two exponential quantities having the same nonzero base, subtract the exponent of the denominator from the exponent of the numerator. Keep in mind that the exponential quantities being divided **must** have the **same base** for this rule to apply.

8 Sample Set C

Find the following quotients. All exponents are natural numbers.

Example 16

$\frac{x^5}{x^2} = \boxed{x^{5-2}} = x^3$ The part in the box is usally done mentally.

Example 17

$$\frac{27a^3b^6c^2}{3a^2bc} = \boxed{9\ a^{3-2}b^{6-1}c^{2-1}} = 9ab^5c$$

Example 18

$$\frac{15x^{\square}}{3x^{\triangle}} = 5x^{\square-\triangle}$$

The bases are the same, so we subtract the exponents. Although we don't know exactly what $\square - \triangle$ is, the notation $\square - \triangle$ indicates the subtraction.

9 Practice Set C

Find each quotient

Exercise 14 *(Solution on p. 11.)*

$\frac{y^9}{y^5}$

Exercise 15 *(Solution on p. 11.)*

$\frac{a^7}{a}$

Exercise 16 *(Solution on p. 11.)*

$\frac{(x+6)^5}{(x+6)^3}$

Exercise 17 *(Solution on p. 11.)*

$\frac{26x^4y^6z^2}{13x^2y^2z}$

When we make the subtraction, $n - m$, in the division $\frac{x^n}{x^m}$, there are three possibilities for the values of the exponents:

1. The exponent of the numerator is greater than the exponent of the denominator, that is, $n > m$. Thus, the exponent, $n - m$, is a natural number.
2. The exponents are the same, that is, $n = m$. Thus, the exponent, $n - m$, is zero, a whole number.
3. The exponent of the denominator is greater than the exponent of the numerator, that is, $n < m$. Thus, the exponent, $n - m$, is an integer.

10 Zero as an Exponent

In Sample Set C, the exponents of the numerators were greater than the exponents of the denominators. Let's study the case when the exponents are the same.

When the exponents are the same, say n , the subtraction $n - n$ produces 0.

Thus, by the second rule of exponents, $\frac{x^n}{x^n} = x^{n-n} = x^0$.

But what real number, if any, does x^0 represent? Let's think for a moment about our experience with division in arithmetic. We know that any nonzero number divided by itself is one.

$\frac{8}{8} = 1, \quad \frac{43}{43} = 1, \quad \frac{258}{258} \quad = 1$

Since the letter x represents some nonzero real number, so does x^n. Thus, $\frac{x^n}{x^n}$

represents some nonzero real number divided by itself. Then $\frac{x^n}{x^n} = 1$.

But we have also established that if $x \neq 0$, $\frac{x^n}{x^n} = x^0$. We now have that $\frac{x^n}{x^n} = x^0$

and $\frac{x^n}{x^n} = 1$. This implies that $x^0 = 1, x \neq 0$.

Exponents can now be natural numbers and zero. We have enlarged our collection of numbers that can be used as exponents from the collection of natural numbers to the collection of whole numbers.

ZERO AS AN EXPONENT

If $x \neq 0, \quad x^0 = 1$

Any number, other than 0, raised to the power of 0, is 1. 0^0 has no meaning (it does not represent a number).

11 Sample Set D

Find each value. Assume the base is not zero.

Example 19

$$6^0 = 1$$

Example 20

$247^0 = 1$

Example 21

$(2a + 5)^0 = 1$

Example 22

$$4y^0 = 4 \cdot 1 = 4$$

Example 23

$$\frac{y^6}{y^6} = y^0 = 1$$

Example 24

$$\frac{2x^2}{x^2} = 2x^0 = 2 \cdot 1 = 2$$

Example 25

$$\begin{aligned} \frac{5(x+4)^8(x-1)^5}{5(x+4)^3(x-1)^5} &= (x+4)^{8-3}(x-1)^{5-5} \\ &= (x+4)^5(x-1)^0 \\ &= (x+4)^5 \end{aligned}$$

12 Practice Set D

Find each value. Assume the base is not zero.

Exercise 18 *(Solution on p. 11.)*

$\frac{y^7}{y^3}$

Exercise 19 *(Solution on p. 11.)*

$\frac{6x^4}{2x^3}$

Exercise 20 *(Solution on p. 11.)*

$\frac{14a^7}{7a^2}$

Exercise 21 *(Solution on p. 11.)*

$\frac{26x^2y^5}{4xy^2}$

Exercise 22 *(Solution on p. 11.)*

$\frac{36a^4b^3c^8}{8ab^3c^6}$

Exercise 23 *(Solution on p. 11.)*

$\frac{51(a-4)^3}{17(a-4)}$

Exercise 24 *(Solution on p. 11.)*

$\frac{52a^7b^3(a+b)^8}{26a^2b(a+b)^8}$

Exercise 25 *(Solution on p. 11.)*

$\frac{a^n}{a^3}$

Exercise 26 *(Solution on p. 12.)*

$\frac{14x^ry^pz^q}{2x^ry^hz^5}$

We will study the case where the exponent of the denominator is greater than the exponent of the numerator in Section here[1].

13 Exercises

Use the product rule and quotient rule of exponents to simplify the following problems. Assume that all bases are nonzero and that all exponents are whole numbers.

Exercise 27 *(Solution on p. 12.)*

$3^2 \cdot 3^3$

Exercise 28

$5^2 \cdot 5^4$

Exercise 29 *(Solution on p. 12.)*

$9^0 \cdot 9^2$

Exercise 30

$7^3 \cdot 7^0$

Exercise 31 *(Solution on p. 12.)*

$2^4 \cdot 2^5$

Exercise 32

x^5x^4

Exercise 33 *(Solution on p. 12.)*

x^2x^3

Exercise 34

a^9a^7

[1]"Basic Operations with Real Numbers: Negative Exponents" <http://cnx.org/content/m21882/latest/>

Exercise 35 *(Solution on p. 12.)*

$y^5 y^7$

Exercise 36

$m^{10} m^2$

Exercise 37 *(Solution on p. 12.)*

$k^8 k^3$

Exercise 38

$y^3 y^4 y^6$

Exercise 39 *(Solution on p. 12.)*

$3x^2 \cdot 2x^5$

Exercise 40

$a^2 a^3 a^8$

Exercise 41 *(Solution on p. 12.)*

$4y^4 \cdot 5y^6$

Exercise 42

$2a^3 b^2 \cdot 3ab$

Exercise 43 *(Solution on p. 12.)*

$12xy^3 z^2 \cdot 4x^2 y^2 z \cdot 3x$

Exercise 44

$(3ab)\left(2a^2 b\right)$

Exercise 45 *(Solution on p. 12.)*

$\left(4x^2\right)\left(8xy^3\right)$

Exercise 46

$(2xy)(3y)\left(4x^2 y^5\right)$

Exercise 47 *(Solution on p. 12.)*

$\left(\frac{1}{4}a^2 b^4\right)\left(\frac{1}{2}b^4\right)$

Exercise 48

$\left(\frac{3}{8}\right)\left(\frac{16}{21}x^2 y^3\right)\left(x^3 y^2\right)$

Exercise 49 *(Solution on p. 12.)*

$\frac{8^5}{8^3}$

Exercise 50

$\frac{6^4}{6^3}$

Exercise 51 *(Solution on p. 12.)*

$\frac{2^9}{2^4}$

Exercise 52

$\frac{4^{16}}{4^{13}}$

Exercise 53 *(Solution on p. 12.)*

$\frac{x^5}{x^3}$

Exercise 54

$\frac{y^4}{y^3}$

Exercise 55 *(Solution on p. 12.)*

$\frac{y^9}{y^4}$

Exercise 56

$\frac{k^{16}}{k^{13}}$

Exercise 57 *(Solution on p. 12.)*

$\frac{x^4}{x^2}$

Exercise 58

$\frac{y^5}{y^2}$

Exercise 59 *(Solution on p. 12.)*

$\frac{m^{16}}{m^9}$

Exercise 60

$\frac{a^9 b^6}{a^5 b^2}$

Exercise 61 *(Solution on p. 12.)*

$\frac{y^3 w^{10}}{y w^5}$

Exercise 62

$\frac{m^{17} n^{12}}{m^{16} n^{10}}$

Exercise 63 *(Solution on p. 12.)*

$\frac{x^5 y^7}{x^3 y^4}$

Exercise 64

$\frac{15x^{20} y^{24} z^4}{5x^{19} yz}$

Exercise 65 *(Solution on p. 12.)*

$\frac{e^{11}}{e^{11}}$

Exercise 66

$\frac{6r^4}{6r^4}$

Exercise 67 *(Solution on p. 12.)*

$\frac{x^0}{x^0}$

Exercise 68

$\frac{a^0 b^0}{c^0}$

Exercise 69 *(Solution on p. 12.)*

$\frac{8a^4 b^0}{4a^3}$

Exercise 70

$\frac{24x^4 y^4 z^0 w^8}{9xyw^7}$

Exercise 71 *(Solution on p. 12.)*

$t^2 \left(y^4\right)$

Exercise 72

$x^3 \left(\frac{x^6}{x^2}\right)$

Exercise 73 *(Solution on p. 12.)*

$a^4 b^6 \left(\frac{a^{10} b^{16}}{a^5 b^7}\right)$

Exercise 74

$3a^2 b^3 \left(\frac{14a^2 b^5}{2b}\right)$

Exercise 75 *(Solution on p. 13.)*

$\frac{(x+3y)^{11}(2x-1)^4}{(x+3y)^3(2x-1)}$

Exercise 76

$\frac{40x^5 z^{10}\left(z-x^4\right)^{12}(x+z)^2}{10z^7(z-x^4)^5}$

Exercise 77 *(Solution on p. 13.)*

$x^n x^r$

Exercise 78

$a^x b^y c^{5z}$

Exercise 79 *(Solution on p. 13.)*

$x^n \cdot x^{n+3}$

Exercise 80

$\frac{x^{n+3}}{x^n}$

Exercise 81 *(Solution on p. 13.)*

$\frac{x^{n+2}x^3}{x^4 x^n}$

Exercise 82

$a^{\bigstar}a^{\bigcirc}$

Exercise 83 *(Solution on p. 13.)*

$m^{\diamond}m^{\bigstar}m^{\triangle}$

Exercise 84

$y^{\triangle}y^{\nabla}$

Exercise 85 *(Solution on p. 13.)*

$a^{\triangle}a^{\nabla}b^{\square}b^{\diamond}$

14 Exercises for Review

Exercise 86

(here[2]) What natural numbers can replace x so that the statement $-5 < x \leq 3$ is true?

Exercise 87 *(Solution on p. 13.)*

(here[3]) Use the distributive property to expand $4x\left(2a+3b\right)$.

Exercise 88

(here[4]) Express $xxxyyyy\left(a+b\right)\left(a+b\right)$ using exponents.

Exercise 89 *(Solution on p. 13.)*

(here[5]) Find the value of $4^2 + 3^2 \cdot 2^3 - 10 \cdot 8$.

Exercise 90

(here[6]) Find the value of $\frac{4^2+(3+2)^2-1}{2^3 \cdot 5} + \frac{2^4\left(3^2-2^3\right)}{4^2}$.

[2]"Basic Properties of Real Numbers: The Real Number Line and the Real Numbers" <http://cnx.org/content/m21895/latest/>

[3]"Basic Properties of Real Numbers: Properties of the Real Numbers" <http://cnx.org/content/m21894/latest/>

[4]"Basic Properties of Real Numbers: Exponents" <http://cnx.org/content/m21883/latest/>

[5]"Basic Properties of Real Numbers: Exponents" <http://cnx.org/content/m21883/latest/>

[6]"Basic Properties of Real Numbers: Exponents" <http://cnx.org/content/m21883/latest/>

Solutions to Exercises in this Module

Solution to Exercise 1 (p. 3)
$x^{2+5} = x^7$
Solution to Exercise 2 (p. 3)
$x^{9+4} = x^{13}$
Solution to Exercise 3 (p. 3)
$y^{6+4} = y^{10}$
Solution to Exercise 4 (p. 3)
$c^{12+8} = c^{20}$
Solution to Exercise 5 (p. 3)
$(x+2)^{3+5} = (x+2)^8$
Solution to Exercise 6 (p. 4)
$6x^7$
Solution to Exercise 7 (p. 4)
$18y^7$
Solution to Exercise 8 (p. 4)
$36a^5b^3$
Solution to Exercise 9 (p. 4)
$56x^6y^8$
Solution to Exercise 10 (p. 4)
$4(x-y)^5$
Solution to Exercise 11 (p. 4)
$8x^8y^7$
Solution to Exercise 12 (p. 4)
$12a^{10}b^5$
Solution to Exercise 13 (p. 4)
a^{n+m+r}
Solution to Exercise 14 (p. 5)
y^4
Solution to Exercise 15 (p. 5)
a^6
Solution to Exercise 16 (p. 5)
$(x+6)^2$
Solution to Exercise 17 (p. 5)
$2x^2y^4z$
Solution to Exercise 18 (p. 7)
$y^{7-3} = y^4$
Solution to Exercise 19 (p. 7)
$3x^{4-3} = 3x$
Solution to Exercise 20 (p. 7)
$2a^{7-2} = 2a^5$
Solution to Exercise 21 (p. 7)
$\frac{13}{2}xy^3$
Solution to Exercise 22 (p. 7)
$\frac{9}{2}a^3c^2$
Solution to Exercise 23 (p. 7)
$3(a-4)^2$
Solution to Exercise 24 (p. 7)
$2a^5b^2$

Solution to Exercise 25 (p. 7)
a^{n-3}
Solution to Exercise 26 (p. 7)
$7y^{p-h}z^{q-5}$
Solution to Exercise 27 (p. 7)
$3^5 = 243$
Solution to Exercise 29 (p. 7)
$9^2 = 81$
Solution to Exercise 31 (p. 7)
$2^9 = 512$
Solution to Exercise 33 (p. 7)
x^5
Solution to Exercise 35 (p. 7)
y^{12}
Solution to Exercise 37 (p. 8)
k^{11}
Solution to Exercise 39 (p. 8)
$6x^7$
Solution to Exercise 41 (p. 8)
$20y^{10}$
Solution to Exercise 43 (p. 8)
$144x^4y^5z^3$
Solution to Exercise 45 (p. 8)
$32x^3y^3$
Solution to Exercise 47 (p. 8)
$\frac{1}{8}a^2b^8$
Solution to Exercise 49 (p. 8)
$8^2 = 64$
Solution to Exercise 51 (p. 8)
$2^5 = 32$
Solution to Exercise 53 (p. 8)
x^2
Solution to Exercise 55 (p. 8)
y^5
Solution to Exercise 57 (p. 8)
x^2
Solution to Exercise 59 (p. 9)
m^7
Solution to Exercise 61 (p. 9)
y^2w^5
Solution to Exercise 63 (p. 9)
x^2y^3
Solution to Exercise 65 (p. 9)
$e^0 = 1$
Solution to Exercise 67 (p. 9)
$x^0 = 1$
Solution to Exercise 69 (p. 9)
$2a$
Solution to Exercise 71 (p. 9)
t^2y^4

Solution to Exercise 73 (p. 9)
$a^9 b^{15}$
Solution to Exercise 75 (p. 9)
$(x+3y)^8 (2x-1)^3$
Solution to Exercise 77 (p. 9)
x^{n+r}
Solution to Exercise 79 (p. 10)
x^{2n+3}
Solution to Exercise 81 (p. 10)
x
Solution to Exercise 83 (p. 10)
$m^{\circ + \star + \triangle}$
Solution to Exercise 85 (p. 10)
$a^{\triangle + \nabla} b^{\square + \diamond}$
Solution to Exercise 87 (p. 10)
$8ax + 12bx$
Solution to Exercise 89 (p. 10)
8