Quadratic Equations: Solving Quadratic Equations Using the Quadratic Formula*

Wade Ellis
Denny Burzynski

This work is produced by The Connexions Project and licensed under the
Creative Commons Attribution License †

Abstract

This module is from Elementary Algebra</link> by Denny Burzynski and Wade Ellis, Jr. Methods of solving quadratic equations as well as the logic underlying each method are discussed. Factoring, extraction of roots, completing the square, and the quadratic formula are carefully developed. The zero-factor property of real numbers is reintroduced. The chapter also includes graphs of quadratic equations based on the standard parabola, y = x^2, and applied problems from the areas of manufacturing, population, physics, geometry, mathematics (numbers and volumes), and astronomy, which are solved using the five-step method. Objectives of this module: recognize the standard form of a quadratic equation, understand the derivation of the quadratic formula, solve quadratic equations using the quadratic formula.

1 Overview

- Standard Form Of A Quadratic Equation
- The Quadratic Formula
- Derivation Of The Quadratic Formula

2 Standard Form Of A Quadratic Equation

We have observed that a quadratic equation is an equation of the form

$$ax^2 + bx + c = 0, \qquad a \neq 0$$

where

a is the coefficient of the quadratic term,
b is the coefficient of the linear term, and
c is the constant term.

Standard Form

The equation $ax^2 + bx + c = 0$ is the **standard form** of a quadratic equation.

*Version 1.4: Jun 1, 2009 11:17 am -0500
†http://creativecommons.org/licenses/by/3.0/

3 Sample Set A

Determine the values of a, b, and c.

Example 1
In the equation $3x^2 + 5x + 2 = 0$,

$$\begin{aligned} a &= 3 \\ b &= 5 \\ c &= 2 \end{aligned}$$

Example 2
In the equation $12x^2 - 2x - 1 = 0$,

$$\begin{aligned} a &= 12 \\ b &= -2 \\ c &= -1 \end{aligned}$$

Example 3
In the equation $2y^2 + 3 = 0$,

$$\begin{aligned} a &= 2 \\ b &= 0 \qquad \text{Because the equation could be written} \\ &\qquad\quad 2y^2 + 0y + 3 = 0 \\ c &= 3 \end{aligned}$$

Example 4
In the equation $-8y^2 + 11y = 0$,

$$\begin{aligned} a &= -8 \\ b &= 11 \\ c &= 0 \qquad \text{Since } -8\text{y}^2 + 11y + 0 = 0. \end{aligned}$$

Example 5
In the equation $z^2 = z + 8$,

$$\begin{aligned} a &= 1 \\ b &= -1 \\ c &= -8 \qquad \text{When we write the equation in standard form, we get } z^2 - z - 8 = 0 \end{aligned}$$

4 Practice Set A

Determine the values of a, b, and c in the following quadratic equations.

Exercise 1 ***(Solution on p. 10.)***
$4x^2 - 3x + 5 = 0$

Exercise 2 ***(Solution on p. 10.)***
$3y^2 - 2y + 9 = 0$

Exercise 3 *(Solution on p. 10.)*
$x^2 - 5x - 1 = 0$

Exercise 4 *(Solution on p. 10.)*
$z^2 - 4 = 0$

Exercise 5 *(Solution on p. 10.)*
$x^2 - 2x = 0$

Exercise 6 *(Solution on p. 10.)*
$y^2 = 5y - 6$

Exercise 7 *(Solution on p. 10.)*
$2x^2 - 4x = -1$

Exercise 8 *(Solution on p. 10.)*
$5x - 3 = -3x^2$

Exercise 9 *(Solution on p. 10.)*
$2x - 11 - 3x^2 = 0$

Exercise 10 *(Solution on p. 10.)*
$y^2 = 0$

The solutions to **all** quadratic equations depend only and completely on the values a, b, and c.

5 The Quadratic Formula

When a quadratic equation is written in standard form so that the values a, b, and c are readily determined, the equation can be solved using the **quadratic formula**. The values that satisfy the equation are found by substituting the values a, b, and c into the formula
Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Keep in mind that the plus or minus symbol, $\pm$, is just a shorthand way of denoting the two possibilities:

$$x = \frac{-b+\sqrt{b^2-4ac}}{2a} \qquad \text{and} \qquad x = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

The quadratic formula can be derived by using the method of completing the square.

6 Derivation Of The Quadratic Formula

Solve $ax^2 + bx + c = 0$ for x by completing the square.

Example 6
Subtract c from both sides.

$ax^2 + bx = -c$

Example 7
Divide both sides by a, the coefficient of x^2.

$x^2 + \frac{b}{a}x = \frac{-c}{a}$

Example 8
Now we have the proper form to complete the square. Take one half the coefficient of x, square it, and add the result to both sides of the equation found in step 2.

(a) $\frac{1}{2} \cdot \frac{b}{a} = \frac{b}{2a}$ is one half the coefficient of x.

(b) $\left(\frac{b}{2a}\right)^2$ is the square of one half the coefficient of x.

$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{-c}{a} + \left(\frac{b}{2a}\right)^2$

Example 9
The left side of the equation is now a perfect square trinomial and can be factored. This gives us

$\left(x + \frac{b}{2a}\right)^2 = \frac{-c}{a} + \frac{b^2}{4a^2}$

Example 10
Add the two fractions on the right side of the equation. The LCD $= 4a^2$.

$$\begin{aligned}
\left(x + \frac{b}{2a}\right)^2 &= \frac{-4ac}{4a^2} + \frac{b^2}{4a^2} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{-4ac+b^2}{4a^2} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2-4ac}{4a^2}
\end{aligned}$$

Example 11
Solve for x using the method of extraction of roots.

$$\begin{aligned}
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2-4ac}{4a^2}} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2-4ac}}{\sqrt{4a^2}} && \sqrt{4a^2} = |2a| = 2|a| = \pm 2a \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2-4ac}}{2a} \\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} && \text{Add these two fractions.} \\
x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}
\end{aligned}$$

7 Sample Set B

Solve each of the following quadratic equations using the quadratic formula.

Example 12
$3x^2 + 5x + 2 = 0$.

1. Identify a, b, and c.

   $a = 3, \quad b = 5, \qquad \text{and} \qquad c = 2$
2. Write the quadratic formula.

   $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$
3. Substitute.

$$\begin{aligned} x &= \frac{-5\pm\sqrt{(5)^2-4(3)(2)}}{2(3)} \\ &= \frac{-5\pm\sqrt{25-24}}{6} \\ &= \frac{-5\pm\sqrt{1}}{6} \\ &= \frac{-5\pm 1}{6} && -5+1=-4 \text{ and } -5-1=-6 \\ &= \frac{-4}{6}, \frac{-6}{6} \\ x &= \frac{-2}{3}, -1 \end{aligned}$$

$Note:$ Since these roots are rational numbers, this equation could have been solved by factoring.

Example 13
$12x^2 - 2x - 1 = 0.$

1. Identify a, b, and c.

   $a = 12, \quad b = -2, \quad \text{and} \quad c = -1$
2. Write the quadratic formula.

   $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$
3. Substitute.

$$\begin{aligned} x &= \frac{-(-2)\pm\sqrt{(-2)^2-4(12)(-1)}}{2(12)} \\ &= \frac{2\pm\sqrt{4+48}}{24} && \text{Simplify.} \\ &= \frac{2\pm\sqrt{52}}{24} && \text{Simplify.} \\ &= \frac{2\pm\sqrt{4\cdot 13}}{24} && \text{Simplify.} \\ &= \frac{2\pm 2\sqrt{13}}{24} && \text{Reduce. Factor 2 from the terms of the numerator.} \\ &= \frac{2\left(1\pm\sqrt{13}\right)}{24} \\ x &= \frac{1\pm\sqrt{13}}{12} \end{aligned}$$

Example 14
$2y^2 + 3 = 0$

1. Identify a, b, and c.

   $a = 2, \quad b = 0, \quad \text{and} \quad c = 3$
2. Write the quadratic formula.

   $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$
3. Substitute.

$$\begin{aligned} x &= \frac{-0\pm\sqrt{0^2-4(2)(3)}}{2(2)} \\ x &= \frac{0\pm\sqrt{-24}}{4} \end{aligned}$$

This equation has no real number solution since we have obtained a negative number under the radical sign.

Example 15
$-8x^2 + 11x = 0$

1. Identify a, b, and c.

   $a = -8, \quad b = 11, \qquad \text{and} \qquad c = 0$
2. Write the quadratic formula.

   $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
3. Substitute.

$$\begin{aligned} x &= \frac{-11 \pm \sqrt{11^2 - 4(-8)(0)}}{2(-8)} \\ &= \frac{-11 \pm \sqrt{121 - 0}}{-16} && \text{Simplify.} \\ &= \frac{-11 \pm \sqrt{121}}{-16} && \text{Simplify.} \\ &= \frac{-11 \pm 11}{-16} \\ x &= 0, \frac{11}{8} \end{aligned}$$

Example 16
$(3x + 1)(x - 4) = x^2 + x - 2$

1. Write the equation in standard form.

$$\begin{aligned} 3x^2 - 11x - 4 &= x^2 + x - 2 \\ 2x^2 - 12x - 2 &= 0 \\ x^2 - 6x - 1 &= 0 \end{aligned}$$
2. Identify a, b, and c.

   $a = 1, \quad b = -6, \qquad \text{and} \qquad c = -1$
3. Write the quadratic formula.

   $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
4. Substitute.

$$\begin{aligned} x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{6 \pm \sqrt{36 + 4}}{2} \\ &= \frac{6 \pm \sqrt{40}}{2} \\ &= \frac{6 \pm \sqrt{4 \cdot 10}}{2} \\ &= \frac{6 \pm 2\sqrt{10}}{2} \\ &= \frac{2\left(3 \pm \sqrt{10}\right)}{2} \\ x &= 3 \pm \sqrt{10} \end{aligned}$$

8 Practice Set B

Solve each of the following quadratic equations using the quadratic formula.

Exercise 11 *(Solution on p. 11.)*
$2x^2 + 3x - 7 = 0$

Exercise 12 *(Solution on p. 11.)*
$5a^2 - 2a - 1 = 0$

Exercise 13 *(Solution on p. 11.)*
$6y^2 + 5 = 0$

Exercise 14 *(Solution on p. 11.)*
$-3m^2 + 2m = 0$

9 Exercises

For the following problems, solve the equations using the quadratic formula.

Exercise 15 *(Solution on p. 11.)*
$x^2 - 2x - 3 = 0$

Exercise 16
$x^2 + 5x + 6 = 0$

Exercise 17 *(Solution on p. 11.)*
$y^2 - 5y + 4 = 0$

Exercise 18
$a^2 + 4a - 21 = 0$

Exercise 19 *(Solution on p. 11.)*
$a^2 + 12a + 20 = 0$

Exercise 20
$b^2 - 4b + 4 = 0$

Exercise 21 *(Solution on p. 11.)*
$b^2 + 4b + 4 = 0$

Exercise 22
$x^2 + 10x + 25 = 0$

Exercise 23 *(Solution on p. 11.)*
$2x^2 - 5x - 3 = 0$

Exercise 24
$6y^2 + y - 2 = 0$

Exercise 25 *(Solution on p. 11.)*
$4x^2 - 2x - 1 = 0$

Exercise 26
$3y^2 + 2y - 1 = 0$

Exercise 27 *(Solution on p. 11.)*
$5a^2 - 2a - 3 = 0$

Exercise 28
$x^2 - 3x + 1 = 0$

Exercise 29 *(Solution on p. 11.)*
$x^2 - 5x - 4 = 0$

Exercise 30
$(x + 2)(x - 1) = 1$

Exercise 31 *(Solution on p. 11.)*
$(a+4)(a-5)=2$

Exercise 32
$(x-3)(x+3)=7$

Exercise 33 *(Solution on p. 11.)*
$(b-4)(b+4)=9$

Exercise 34
$x^2+8x=2$

Exercise 35 *(Solution on p. 11.)*
$y^2=-5y+4$

Exercise 36
$x^2=-3x+7$

Exercise 37 *(Solution on p. 11.)*
$x^2=-2x-1$

Exercise 38
$x^2+x+1=0$

Exercise 39 *(Solution on p. 11.)*
$a^2+3a-4=0$

Exercise 40
$y^2+y=-4$

Exercise 41 *(Solution on p. 11.)*
$b^2+3b=-2$

Exercise 42
$x^2+6x+8=-x-2$

Exercise 43 *(Solution on p. 11.)*
$x^2+4x=2x-5$

Exercise 44
$6b^2+5b-4=b^2+b+1$

Exercise 45 *(Solution on p. 11.)*
$4a^2+7a-2=-2a+a$

Exercise 46
$(2x+5)(x-4)=x^2-x+2$

Exercise 47 *(Solution on p. 11.)*
$(x-4)^2=3$

Exercise 48
$(x+2)^2=4$

Exercise 49 *(Solution on p. 11.)*
$(b-6)^2=8$

Exercise 50
$(3-x)^2=6$

Exercise 51 *(Solution on p. 11.)*
$3\left(x^2+1\right)=2(x+7)$

Exercise 52
$2\left(y^2-3\right)=-3(y-1)$

Exercise 53 *(Solution on p. 12.)*

$-4\left(a^2+2\right)+3=5$

Exercise 54

$-\left(x^2+3x-1\right)=2$

10 Exercises For Review

Exercise 55 *(Solution on p. 12.)*

(**here**[1]) Simplify $\left(\frac{x^8y^7z^5}{x^4y^6z^2}\right)^2$.

Exercise 56

(**here**[2]) Write $4a^{-6}b^2c^3a^5b^{-3}$ so that only positive exponents appear.

Exercise 57 *(Solution on p. 12.)*

(**here**[3]) Find the product: $(2y+7)(3y-1)$.

Exercise 58

(**here**[4]) Simplify: $\sqrt{80}-\sqrt{45}$.

Exercise 59 *(Solution on p. 12.)*

(**here**[5]) Solve $x^2-4x-12=0$ by completing the square.

[1] "Basic Properties of Real Numbers: The Power Rules for Exponents" <http://cnx.org/content/m21897/latest/>

[2] "Basic Operations with Real Numbers: Negative Exponents" <http://cnx.org/content/m21882/latest/>

[3] "Algebraic Expressions and Equations: Combining Polynomials Using Multiplication" <http://cnx.org/content/m21852/latest/>

[4] "Roots, Radicals, and Square Root Equations: Addition and Subtraction of Square Root Expressions" <http://cnx.org/content/m21957/latest/>

[5] "Quadratic Equations: Solving Quadratic Equations Using the Method of Completing the Square" <http://cnx.org/content/m21935/latest/>

Solutions to Exercises in this Module

Solution to Exercise (p. 2)

$$\begin{aligned} a &= 4 \\ b &= -3 \\ c &= 5 \end{aligned}$$

Solution to Exercise (p. 2)

$$\begin{aligned} a &= 3 \\ b &= -2 \\ c &= 9 \end{aligned}$$

Solution to Exercise (p. 3)

$$\begin{aligned} a &= 1 \\ b &= -5 \\ c &= -1 \end{aligned}$$

Solution to Exercise (p. 3)

$$\begin{aligned} a &= 1 \\ b &= 0 \\ c &= -4 \end{aligned}$$

Solution to Exercise (p. 3)

$$\begin{aligned} a &= 1 \\ b &= -2 \\ c &= 0 \end{aligned}$$

Solution to Exercise (p. 3)

$$\begin{aligned} a &= 1 \\ b &= -5 \\ c &= 6 \end{aligned}$$

Solution to Exercise (p. 3)

$$\begin{aligned} a &= 2 \\ b &= -4 \\ c &= 1 \end{aligned}$$

Solution to Exercise (p. 3)

$$\begin{aligned} a &= 3 \\ b &= 5 \\ c &= -3 \end{aligned}$$

Solution to Exercise (p. 3)

$$\begin{aligned} a &= -3 \\ b &= 2 \\ c &= -11 \end{aligned}$$

Solution to Exercise (p. 3)

$$\begin{aligned} a &= 1 \\ b &= 0 \\ c &= 0 \end{aligned}$$

Solution to Exercise (p. 6)
$x = \frac{-3 \pm \sqrt{65}}{4}$
Solution to Exercise (p. 7)
$a = \frac{1 \pm \sqrt{6}}{5}$
Solution to Exercise (p. 7)
no real number solution
Solution to Exercise (p. 7)
$m = 0, \frac{2}{3}$
Solution to Exercise (p. 7)
$x = 3, \ -1$
Solution to Exercise (p. 7)
$y = 1, 4$
Solution to Exercise (p. 7)
$a = -2, \ -10$
Solution to Exercise (p. 7)
$b = -2$
Solution to Exercise (p. 7)
$x = 3, -\frac{1}{2}$
Solution to Exercise (p. 7)
$x = \frac{1 \pm \sqrt{5}}{4}$
Solution to Exercise (p. 7)
$a = 1, \ -\frac{3}{5}$
Solution to Exercise (p. 7)
$x = \frac{5 \pm \sqrt{41}}{2}$
Solution to Exercise (p. 7)
$a = \frac{1 \pm \sqrt{89}}{2}$
Solution to Exercise (p. 8)
$b = \pm 5$
Solution to Exercise (p. 8)
$y = \frac{-5 \pm \sqrt{41}}{2}$
Solution to Exercise (p. 8)
$x = -1$
Solution to Exercise (p. 8)
$a = -4, 1$
Solution to Exercise (p. 8)
$b = -1, \ -2$
Solution to Exercise (p. 8)
No real number solution.
Solution to Exercise (p. 8)
$\frac{-2 \pm \sqrt{6}}{2}$
Solution to Exercise (p. 8)
$x = 4 \pm \sqrt{3}$
Solution to Exercise (p. 8)
$b = 6 \pm 2\sqrt{2}$

Solution to Exercise (p. 8)
$x = \frac{1 \pm \sqrt{34}}{3}$
Solution to Exercise (p. 8)
No real number solution.
Solution to Exercise (p. 9)
$x^8 y^2 z^6$
Solution to Exercise (p. 9)
$6y^2 + 19y - 7$
Solution to Exercise (p. 9)
$x = -2,\ 6$