Roots, Radicals, and Square Root Equations: Addition and Subtraction of Square Root Expressions*

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Abstract

This module is from Elementary Algebra by Denny Burzynski and Wade Ellis, Jr. The distinction between the principal square root of the number x and the secondary square root of the number x is made by explanation and by example. The simplification of the radical expressions that both involve and do not involve fractions is shown in many detailed examples; this is followed by an explanation of how and why radicals are eliminated from the denominator of a radical expression. Real-life applications of radical equations have been included, such as problems involving daily output, daily sales, electronic resonance frequency, and kinetic energy. Objectives of this module: understand the process used in adding and subtracting square roots, be able to add and subtract square roots.

1 Overview

- The Logic Behind The Process
- The Process

2 The Logic Behind The Process

Now we will study methods of simplifying radical expressions such as

$$4\sqrt{3}+8\sqrt{3} \qquad \text{or} \qquad 5\sqrt{2x}-11\sqrt{2x}+4\left(\sqrt{2x}+1\right)$$

The procedure for adding and subtracting square root expressions will become apparent if we think back to the procedure we used for simplifying polynomial expressions such as

$$4x+8x \qquad \text{or} \qquad 5a-11a+4\left(a+1\right)$$

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The variables x and a are letters representing some unknown quantities (perhaps x represents $\sqrt{3}$ and a represents $\sqrt{2x}$). Combining like terms gives us

$$4x + 8x = 12x \qquad \text{or} \qquad 4\sqrt{3} + 8\sqrt{3} = 12\sqrt{3}$$

and

$$\begin{array}{ccc} 5a - 11a + 4(a+1) & \text{or} & 5\sqrt{2x} - 11\sqrt{2x} + 4\left(\sqrt{2x}+1\right) \\ 5a - 11a + 4a + 4 & & 5\sqrt{2x} - 11\sqrt{2x} + 4\sqrt{2x} + 4 \\ -2a + 4 & & -2\sqrt{2x} + 4 \end{array}$$

3 The Process

Let's consider the expression $4\sqrt{3} + 8\sqrt{3}$. There are two ways to look at the simplification process:

1. We are asking, "How many square roots of 3 do we have?"

   $4\sqrt{3}$ means we have 4 "square roots of 3"

   $8\sqrt{3}$ means we have 8 "square roots of 3"

   Thus, altogether we have 12 "square roots of 3."
2. We can also use the idea of combining like terms. If we recall, the process of combining like terms is based on the distributive property

   $$4x + 8x = 12x \qquad \text{because} \qquad 4x + 8x = (4+8)\,x = 12x$$

   We could simplify $4\sqrt{3} + 8\sqrt{3}$ using the distributive property.

   $$4\sqrt{3} + 8\sqrt{3} = (4+8)\sqrt{3} = 12\sqrt{3}$$

Both methods will give us the same result. The first method is probably a bit quicker, but keep in mind, however, that the process works because it is based on one of the basic rules of algebra, the distributive property of real numbers.

4 Sample Set A

Simplify the following radical expressions.

Example 1
$-6\sqrt{10} + 11\sqrt{10} = 5\sqrt{10}$

Example 2
$4\sqrt{32} + 5\sqrt{2}$. Simplify $\sqrt{32}$.

$$\begin{aligned} 4\sqrt{16\cdot 2} + 5\sqrt{2} &= 4\sqrt{16}\sqrt{2} + 5\sqrt{2} \\ &= 4\cdot 4\sqrt{2} + 5\sqrt{2} \\ &= 16\sqrt{2} + 5\sqrt{2} \\ &= 21\sqrt{2} \end{aligned}$$

Example 3

$-3x\sqrt{75}+2x\sqrt{48}-x\sqrt{27}.$ Simple each of the three radicals.

$$\begin{aligned} &= -3x\sqrt{25\cdot 3}+2x\sqrt{16\cdot 3}-x\sqrt{9\cdot 3} \\ &= -15x\sqrt{3}+8x\sqrt{3}-3x\sqrt{3} \\ &= (-15x+8x-3x)\sqrt{3} \\ &= -10x\sqrt{3} \end{aligned}$$

Example 4

$5a\sqrt{24a^3}-7\sqrt{54a^5}+a^2\sqrt{6a}+6a.$ Simplify each radical.

$$\begin{aligned} &= 5a\sqrt{4\cdot 6\cdot a^2\cdot a}-7\sqrt{9\cdot 6\cdot a^4\cdot a}+a^2\sqrt{6a}+6a \\ &= 10a^2\sqrt{6a}-21a^2\sqrt{6a}+a^2\sqrt{6a}+6a \\ &= \left(10a^2-21a^2+a^2\right)\sqrt{6a}+6a \\ &= -10a^2\sqrt{6a}+6a \\ &= -2a\left(5a\sqrt{6a}-3\right) \end{aligned}$$

Factor out $-2a$. (This step is optional.)

5 Practice Set A

Find each sum or difference.

Exercise 1 *(Solution on p. 10.)*

$4\sqrt{18}-5\sqrt{8}$

Exercise 2 *(Solution on p. 10.)*

$6x\sqrt{48}+8x\sqrt{75}$

Exercise 3 *(Solution on p. 10.)*

$-7\sqrt{84x}-12\sqrt{189x}+2\sqrt{21x}$

Exercise 4 *(Solution on p. 10.)*

$9\sqrt{6}-8\sqrt{6}+3$

Exercise 5 *(Solution on p. 10.)*

$\sqrt{a^3}+4a\sqrt{a}$

Exercise 6 *(Solution on p. 10.)*

$4x\sqrt{54x^3}+\sqrt{36x^2}+3\sqrt{24x^5}-3x$

6 Sample Set B

Example 5

$\sqrt{7}(\sqrt{8}-3).$ **Use the distributive property to distribute the $\sqrt{7}$.**

$$\begin{aligned} \sqrt{7}(\sqrt{8}-3) &= \sqrt{7}\,\sqrt{8}-\sqrt{7}\cdot 3 \\ &= \sqrt{56}-3\sqrt{7} \\ &= \sqrt{4\cdot 14}-3\sqrt{7} \\ &= 2\sqrt{14}-3\sqrt{7} \end{aligned}$$

Simply $\sqrt{56}$.

Example 6

$(\sqrt{2}+\sqrt{3})(\sqrt{5}+\sqrt{12})$. **Use the distributive property.**

$$(\sqrt{2}+\sqrt{3})(\sqrt{5}+\sqrt{12}) = \sqrt{2}\,\sqrt{5}+\sqrt{2}\,\sqrt{12}+\sqrt{3}\,\sqrt{5}+\sqrt{3}\,\sqrt{12}$$
$$= \sqrt{10}+\sqrt{24}+\sqrt{15}+\sqrt{36}$$
$$= \sqrt{10}+2\sqrt{6}+\sqrt{15}+6$$

Example 7

$$(4\sqrt{2}-3\sqrt{6})(5\sqrt{2}+\sqrt{6})$$
$$(4\sqrt{2}-3\sqrt{6})(5\sqrt{2}+\sqrt{6})$$
$$= 4\cdot 5\sqrt{2}\,\sqrt{2}+4\sqrt{2}\,\sqrt{6}-3\cdot 5\sqrt{6}\,\sqrt{2}-3\sqrt{6}\,\sqrt{6}$$
$$= 20\sqrt{4}+4\sqrt{12}-15\sqrt{12}-3\sqrt{36}$$
$$= 20\cdot 2+4\sqrt{4\cdot 3}-15\sqrt{4\cdot 3}-3\cdot 6$$
$$= 40+8\sqrt{3}-30\sqrt{3}-18$$
$$= 22-22\sqrt{3} \qquad \text{or} \qquad 22(1-\sqrt{3})$$

Example 8

$\frac{3+\sqrt{8}}{3-\sqrt{8}}$.

We'll rationalize the denominator by multiplying this fraction by 1 in the form $\frac{3+\sqrt{8}}{3+\sqrt{8}}$.

$$\begin{aligned}
\frac{3+\sqrt{8}}{3-\sqrt{8}}\cdot\frac{3+\sqrt{8}}{3+\sqrt{8}} &= \frac{\left(3+\sqrt{8}\right)\left(3+\sqrt{8}\right)}{3^2-\left(\sqrt{8}\right)^2}\\
&= \frac{9+3\sqrt{8}+3\sqrt{8}+\sqrt{8}\sqrt{8}}{9-8}\\
&= \frac{9+6\sqrt{8}+8}{1}\\
&= 17+6\sqrt{8}\\
&= 17+6\sqrt{4\cdot 2}\\
&= 17+12\sqrt{2}
\end{aligned}$$

Example 9

$\frac{2+\sqrt{7}}{4-\sqrt{3}}$.

Rationalize the denominator by multiplying this fraction by 1 in the from $\frac{4+\sqrt{3}}{4+\sqrt{3}}$.

$$\begin{aligned}\frac{2+\sqrt{7}}{4-\sqrt{3}}\cdot\frac{4+\sqrt{3}}{4+\sqrt{3}} &= \frac{\left(2+\sqrt{7}\right)\left(4+\sqrt{3}\right)}{4^2-\left(\sqrt{3}\right)^2}\\ &= \frac{8+2\sqrt{3}+4\sqrt{7}+\sqrt{21}}{16-3}\\ &= \frac{8+2\sqrt{3}+4\sqrt{7}+\sqrt{21}}{13}\end{aligned}$$

7 Practice Set B

Simplify each by performing the indicated operation.

Exercise 7 *(Solution on p. 10.)*

$\sqrt{5}\left(\sqrt{6}-4\right)$

Exercise 8 *(Solution on p. 10.)*

$\left(\sqrt{5}+\sqrt{7}\right)\left(\sqrt{2}+\sqrt{8}\right)$

Exercise 9 *(Solution on p. 10.)*

$\left(3\sqrt{2}-2\sqrt{3}\right)\left(4\sqrt{3}+\sqrt{8}\right)$

Exercise 10 *(Solution on p. 10.)*

$\frac{4+\sqrt{5}}{3-\sqrt{8}}$

8 Exercises

For the following problems, simplify each expression by performing the indicated operation.

Exercise 11 *(Solution on p. 10.)*

$4\sqrt{5}-2\sqrt{5}$

Exercise 12

$10\sqrt{2}+8\sqrt{2}$

Exercise 13 *(Solution on p. 10.)*

$-3\sqrt{6}-12\sqrt{6}$

Exercise 14

$-\sqrt{10}-2\sqrt{10}$

Exercise 15 *(Solution on p. 10.)*

$3\sqrt{7x}+2\sqrt{7x}$

Exercise 16

$6\sqrt{3a}+\sqrt{3a}$

Exercise 17 *(Solution on p. 10.)*

$2\sqrt{18}+5\sqrt{32}$

Exercise 18

$4\sqrt{27}-3\sqrt{48}$

Exercise 19 *(Solution on p. 10.)*

$\sqrt{200}-\sqrt{128}$

Exercise 20

$4\sqrt{300}+2\sqrt{500}$

Exercise 21 *(Solution on p. 10.)*
$6\sqrt{40}+8\sqrt{80}$

Exercise 22
$2\sqrt{120}-5\sqrt{30}$

Exercise 23 *(Solution on p. 10.)*
$8\sqrt{60}-3\sqrt{15}$

Exercise 24
$\sqrt{a^3}-3a\sqrt{a}$

Exercise 25 *(Solution on p. 10.)*
$\sqrt{4x^3}+x\sqrt{x}$

Exercise 26
$2b\sqrt{a^3b^5}+6a\sqrt{ab^7}$

Exercise 27 *(Solution on p. 10.)*
$5xy\sqrt{2xy^3}-3y^2\sqrt{2x^3y}$

Exercise 28
$5\sqrt{20}+3\sqrt{45}-3\sqrt{40}$

Exercise 29 *(Solution on p. 10.)*
$\sqrt{24}-2\sqrt{54}-4\sqrt{12}$

Exercise 30
$6\sqrt{18}+5\sqrt{32}+4\sqrt{50}$

Exercise 31 *(Solution on p. 10.)*
$-8\sqrt{20}-9\sqrt{125}+10\sqrt{180}$

Exercise 32
$2\sqrt{27}+4\sqrt{3}-6\sqrt{12}$

Exercise 33 *(Solution on p. 10.)*
$\sqrt{14}+2\sqrt{56}-3\sqrt{136}$

Exercise 34
$3\sqrt{2}+2\sqrt{63}+5\sqrt{7}$

Exercise 35 *(Solution on p. 10.)*
$4ax\sqrt{3x}+2\sqrt{3a^2x^3}+7\sqrt{3a^2x^3}$

Exercise 36
$3by\sqrt{5y}+4\sqrt{5b^2y^3}-2\sqrt{5b^2y^3}$

Exercise 37 *(Solution on p. 10.)*
$\sqrt{2}\left(\sqrt{3}+1\right)$

Exercise 38
$\sqrt{3}\left(\sqrt{5}-3\right)$

Exercise 39 *(Solution on p. 10.)*
$\sqrt{5}\left(\sqrt{3}-\sqrt{2}\right)$

Exercise 40
$\sqrt{7}\left(\sqrt{6}-\sqrt{3}\right)$

Exercise 41 *(Solution on p. 11.)*
$\sqrt{8}\left(\sqrt{3}+\sqrt{2}\right)$

Exercise 42
$\sqrt{10}\left(\sqrt{10}-\sqrt{5}\right)$

Exercise 43 *(Solution on p. 11.)*
$\left(1+\sqrt{3}\right)\left(2-\sqrt{3}\right)$

Exercise 44
$\left(5+\sqrt{6}\right)\left(4-\sqrt{6}\right)$

Exercise 45 *(Solution on p. 11.)*
$\left(3-\sqrt{2}\right)\left(4-\sqrt{2}\right)$

Exercise 46
$\left(5+\sqrt{7}\right)\left(4-\sqrt{7}\right)$

Exercise 47 *(Solution on p. 11.)*
$\left(\sqrt{2}+\sqrt{5}\right)\left(\sqrt{2}+3\sqrt{5}\right)$

Exercise 48
$\left(2\sqrt{6}-\sqrt{3}\right)\left(3\sqrt{6}+2\sqrt{3}\right)$

Exercise 49 *(Solution on p. 11.)*
$\left(4\sqrt{5}-2\sqrt{3}\right)\left(3\sqrt{5}+\sqrt{3}\right)$

Exercise 50
$\left(3\sqrt{8}-2\sqrt{2}\right)\left(4\sqrt{2}-5\sqrt{8}\right)$

Exercise 51 *(Solution on p. 11.)*
$\left(\sqrt{12}+5\sqrt{3}\right)\left(2\sqrt{3}-2\sqrt{12}\right)$

Exercise 52
$\left(1+\sqrt{3}\right)^2$

Exercise 53 *(Solution on p. 11.)*
$\left(3+\sqrt{5}\right)^2$

Exercise 54
$\left(2-\sqrt{6}\right)^2$

Exercise 55 *(Solution on p. 11.)*
$\left(2-\sqrt{7}\right)^2$

Exercise 56
$\left(1+\sqrt{3x}\right)^2$

Exercise 57 *(Solution on p. 11.)*
$\left(2+\sqrt{5x}\right)^2$

Exercise 58
$\left(3-\sqrt{3x}\right)^2$

Exercise 59 *(Solution on p. 11.)*
$\left(8-\sqrt{6b}\right)^2$

Exercise 60
$\left(2a+\sqrt{5a}\right)^2$

Exercise 61 *(Solution on p. 11.)*
$\left(3y-\sqrt{7y}\right)^2$

Exercise 62
$\left(3+\sqrt{3}\right)\left(3-\sqrt{3}\right)$

Exercise 63 *(Solution on p. 11.)*
$\left(2+\sqrt{5}\right)\left(2-\sqrt{5}\right)$

Exercise 64
$\left(8+\sqrt{10}\right)\left(8-\sqrt{10}\right)$

Exercise 65 *(Solution on p. 11.)*
$\left(6+\sqrt{7}\right)\left(6-\sqrt{7}\right)$

Exercise 66
$\left(\sqrt{2}+\sqrt{3}\right)\left(\sqrt{2}-\sqrt{3}\right)$

Exercise 67 *(Solution on p. 11.)*
$\left(\sqrt{5}+\sqrt{2}\right)\left(\sqrt{5}-\sqrt{2}\right)$

Exercise 68
$\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)$

Exercise 69 *(Solution on p. 11.)*
$\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-\sqrt{y}\right)$

Exercise 70
$\frac{2}{5+\sqrt{3}}$

Exercise 71 *(Solution on p. 11.)*
$\frac{4}{6+\sqrt{2}}$

Exercise 72
$\frac{1}{3-\sqrt{2}}$

Exercise 73 *(Solution on p. 11.)*
$\frac{1}{4-\sqrt{3}}$

Exercise 74
$\frac{8}{2-\sqrt{6}}$

Exercise 75 *(Solution on p. 11.)*
$\frac{2}{3-\sqrt{7}}$

Exercise 76
$\frac{\sqrt{5}}{3+\sqrt{3}}$

Exercise 77 *(Solution on p. 11.)*
$\frac{\sqrt{3}}{6+\sqrt{6}}$

Exercise 78
$\frac{2-\sqrt{8}}{2+\sqrt{8}}$

Exercise 79 *(Solution on p. 11.)*
$\frac{4+\sqrt{5}}{4-\sqrt{5}}$

Exercise 80
$\frac{1+\sqrt{6}}{1-\sqrt{6}}$

Exercise 81 *(Solution on p. 11.)*
$\frac{8-\sqrt{3}}{2+\sqrt{18}}$

Exercise 82
$\frac{6-\sqrt{2}}{4+\sqrt{12}}$

Exercise 83 *(Solution on p. 11.)*
$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

Exercise 84
$\frac{\sqrt{6a}-\sqrt{8a}}{\sqrt{8a}+\sqrt{6a}}$

Exercise 85 *(Solution on p. 11.)*
$\frac{\sqrt{2b}-\sqrt{3b}}{\sqrt{3b}+\sqrt{2b}}$

9 Exercises For Review

Exercise 86

(**here**[1]) Simplify $\left(\frac{x^5y^3}{x^2y}\right)^5$.

Exercise 87 ***(Solution on p. 11.)***

(**here**[2]) Simplify $\left(8x^3y\right)^2\left(x^2y^3\right)^4$.

Exercise 88

(**here**[3]) Write $(x-1)^4(x-1)^{-7}$ so that only positive exponents appear.

Exercise 89 ***(Solution on p. 11.)***

(**here**[4]) Simplify $\sqrt{27x^5y^{10}z^3}$.

Exercise 90

(**here**[5]) Simplify $\frac{1}{2+\sqrt{x}}$ by rationalizing the denominator.

[1] "Basic Properties of Real Numbers: The Power Rules for Exponents" <http://cnx.org/content/m21897/latest/>
[2] "Basic Properties of Real Numbers: The Power Rules for Exponents" <http://cnx.org/content/m21897/latest/>
[3] "Basic Operations with Real Numbers: Negative Exponents" <http://cnx.org/content/m21882/latest/>
[4] "Roots, Radicals, and Square Root Equations: Simplifying Square Root Expressions" <http://cnx.org/content/m21973/latest/>
[5] "Roots, Radicals, and Square Root Equations: Division of Square Root Expressions" <http://cnx.org/content/m21968/latest/>

Solutions to Exercises in this Module

Solution to Exercise (p. 3)
$2\sqrt{2}$
Solution to Exercise (p. 3)
$64x\sqrt{3}$
Solution to Exercise (p. 3)
$-48\sqrt{21x}$
Solution to Exercise (p. 3)
$\sqrt{6}+3$
Solution to Exercise (p. 3)
$5a\sqrt{a}$
Solution to Exercise (p. 3)
$18x^2\sqrt{6x}+3x$
Solution to Exercise (p. 5)
$\sqrt{30}-4\sqrt{5}$
Solution to Exercise (p. 5)
$3\sqrt{10}+3\sqrt{14}$
Solution to Exercise (p. 5)
$8\sqrt{6}-12$
Solution to Exercise (p. 5)
$12+8\sqrt{2}+3\sqrt{5}+2\sqrt{10}$
Solution to Exercise (p. 5)
$2\sqrt{5}$
Solution to Exercise (p. 5)
$-15\sqrt{6}$
Solution to Exercise (p. 5)
$5\sqrt{7x}$
Solution to Exercise (p. 5)
$26\sqrt{2}$
Solution to Exercise (p. 5)
$2\sqrt{2}$
Solution to Exercise (p. 5)
$12\sqrt{10}+32\sqrt{5}$
Solution to Exercise (p. 6)
$13\sqrt{15}$
Solution to Exercise (p. 6)
$3x\sqrt{x}$
Solution to Exercise (p. 6)
$2xy^2\sqrt{2xy}$
Solution to Exercise (p. 6)
$-4\sqrt{6}-8\sqrt{3}$
Solution to Exercise (p. 6)
$-\sqrt{5}$
Solution to Exercise (p. 6)
$5\sqrt{14}-6\sqrt{34}$
Solution to Exercise (p. 6)
$13ax\sqrt{3x}$
Solution to Exercise (p. 6)
$\sqrt{6}+\sqrt{2}$

Solution to Exercise (p. 6)
$\sqrt{15}-\sqrt{10}$
Solution to Exercise (p. 6)
$2\left(\sqrt{6}+2\right)$
Solution to Exercise (p. 6)
$-1+\sqrt{3}$
Solution to Exercise (p. 7)
$7\left(2-\sqrt{2}\right)$
Solution to Exercise (p. 7)
$17+4\sqrt{10}$
Solution to Exercise (p. 7)
$54-2\sqrt{15}$
Solution to Exercise (p. 7)
-42
Solution to Exercise (p. 7)
$14+6\sqrt{5}$
Solution to Exercise (p. 7)
$11-4\sqrt{7}$
Solution to Exercise (p. 7)
$4+4\sqrt{5x}+5x$
Solution to Exercise (p. 7)
$64-16\sqrt{6b}+6b$
Solution to Exercise (p. 7)
$9y^2-6y\sqrt{7y}+7y$
Solution to Exercise (p. 7)
-1
Solution to Exercise (p. 7)
29
Solution to Exercise (p. 8)
3
Solution to Exercise (p. 8)
$x-y$
Solution to Exercise (p. 8)
$\frac{2\left(6-\sqrt{2}\right)}{17}$
Solution to Exercise (p. 8)
$\frac{4+\sqrt{3}}{13}$
Solution to Exercise (p. 8)
$3+\sqrt{7}$
Solution to Exercise (p. 8)
$\frac{2\sqrt{3}-\sqrt{2}}{10}$
Solution to Exercise (p. 8)
$\frac{21+8\sqrt{5}}{11}$
Solution to Exercise (p. 8)
$\frac{-16+2\sqrt{3}+24\sqrt{2}-3\sqrt{6}}{14}$
Solution to Exercise (p. 8)
$5-2\sqrt{6}$
Solution to Exercise (p. 8)
$2\sqrt{6}-5$
Solution to Exercise (p. 9)
$64x^{14}y^{14}$

Solution to Exercise (p. 9)

$3x^2y^5z\sqrt{3xz}$