Rational Expressions: Multiplying and Dividing Rational Expressions*

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Abstract

<para>This module is from <link document="col10614">Elementary Algebra</link> by Denny Burzynski and Wade Ellis, Jr.</para> <para>A detailed study of arithmetic operations with rational expressions is presented in this chapter, beginning with the definition of a rational expression and then proceeding immediately to a discussion of the domain. The process of reducing a rational expression and illustrations of multiplying, dividing, adding, and subtracting rational expressions are also included. Since the operations of addition and subtraction can cause the most difficulty, they are given particular attention. We have tried to make the written explanation of the examples clearer by using a "freeze frame" approach, which walks the student through the operation step by step.</para> <para>The five-step method of solving applied problems is included in this chapter to show the problem-solving approach to number problems, work problems, and geometry problems. The chapter also illustrates simplification of complex rational expressions, using the combine-divide method and the LCD-multiply-divide method.</para> <para>Objectives of this module: be able to multiply and divide rational expressions.</para>

1 Overview

- Multiplication Of Rational Expressions
- Division Of Rational Expressions

2 Multiplication Of Rational Expressions

Rational expressions are multiplied together in much the same way that arithmetic fractions are multiplied together. To multiply rational numbers, we do the following:

Definition 1: Method for Multiplying Rational Numbers

1. Reduce each fraction to lowest terms.
2. Multiply the numerators together.
3. Multiply the denominators together.

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Rational expressions are multiplied together using exactly the same three steps. Since rational expressions tend to be longer than arithmetic fractions, we can simplify the multiplication process by adding one more step.

Definition 2: Method for Multiplying Rational Expressions

1. Factor all numerators and denominators.
2. Reduce to lowest terms first by dividing out all common factors. (It is perfectly legitimate to cancel the numerator of one fraction with the denominator of another.)
3. Multiply numerators together.
4. Multiply denominators. It is often convenient, but not necessary, to leave denominators in factored form.

3 Sample Set A

Perform the following multiplications.

Example 1

$$\frac{3}{4}\cdot\frac{1}{2}=\frac{3\cdot 1}{4\cdot 2}=\frac{3}{8}$$

Example 2

$$\frac{8}{9}\cdot\frac{1}{6}=\frac{\overset{4}{\cancel{8}}}{9}\cdot\frac{1}{\underset{3}{\cancel{6}}}=\frac{4\cdot 1}{9\cdot 3}=\frac{4}{27}$$

Example 3

$$\frac{3x}{5y}\cdot\frac{7}{12y}=\frac{\overset{1}{\cancel{3}}\,x}{5y}\cdot\frac{7}{\underset{4}{\cancel{12}}y}=\frac{x\cdot 7}{5y\cdot 4y}=\frac{7x}{20y^2}$$

Example 4

$\frac{x+4}{x-2}\cdot\frac{x+7}{x+4}$	Divide out the common factor $x+4$.
$\frac{\cancel{x+4}}{x-2}\cdot\frac{x+7}{\cancel{x+4}}$	Multiply numerators and denominators together.
$\frac{x+7}{x-2}$	

Example 5

$\frac{x^2+x-6}{x^2-4x+3}\cdot\frac{x^2-2x-3}{x^2+4x-12}\cdot$	Factor.
$\frac{(x+3)(x-2)}{(x-3)(x-1)}\cdot\frac{(x-3)(x+1)}{(x+6)(x-2)}$	Divide out the common factors $x-2$ and $x-3$.
$\frac{(x+3)\cancel{(x-2)}}{\cancel{(x-3)}(x-1)}\cdot\frac{\cancel{(x-3)}(x+1)}{(x+6)\cancel{(x-2)}}$	Multiply.
$\frac{(x+3)(x+1)}{(x-1)(x+6)}$ or $\frac{x^2+4x+3}{(x-1)(x+6)}$ or $\frac{x^2+4x+3}{x^2+5x-6}$	

Each of these three forms is an acceptable form of the same answer.

Example 6

$\frac{2x+6}{8x-16} \cdot \frac{x^2-4}{x^2-x-12}$.	Factor.
$\frac{2(x+3)}{8(x-2)} \cdot \frac{(x+2)(x-2)}{(x-4)(x+3)}$	Divide out the common factors $2, x+3$ and $x-2$.
$\frac{\overset{1}{\cancel{2}}(x+3)}{\underset{4}{\cancel{8}}(x-2)} \cdot \frac{(x+2)\cancel{(x-2)}}{\cancel{(x+3)}(x-4)}$	Multiply.
$\frac{x+2}{4(x-4)}$ or $\frac{x+2}{4x-16}$	

Both these forms are acceptable forms of the same answer.

Example 7

$3x^2 \cdot \frac{x+7}{x-5}$.	Rewrite $3x^2$ as $\frac{3x^2}{1}$.
$\frac{3x^2}{1} \cdot \frac{x+7}{x-5}$	Multiply.
$\frac{3x^2(x+7)}{x-5}$	

Example 8

$$(x-3) \cdot \frac{4x-9}{x^2-6x+9}.$$
$$\frac{\cancel{(x-3)}}{1} \cdot \frac{4x-9}{\cancel{(x-3)}(x-3)}$$
$$\frac{4x-9}{x-3}$$

Example 9

$\frac{-x^2-3x-2}{x^2+8x+15} \cdot \frac{4x+20}{x^2+2x}$.	Factor -1 from the first numerator.
$\frac{-\left(x^2+3x+2\right)}{x^2+8x+15} \cdot \frac{4x+20}{x^2+2x}$	Factor.
$\frac{-(x+1)\cancel{(x+2)}}{(x+3)\cancel{(x+5)}} \cdot \frac{4\cancel{(x+5)}}{x\cancel{(x+2)}}$	Multiply.
$\frac{-4(x+1)}{x(x+3)} = \frac{-4x-1}{x(x+3)}$ or $\frac{-4x-1}{x^2+3x}$	

4 Practice Set A

Perform each multiplication.

Exercise 1 *(Solution on p. 10.)*

$\frac{5}{3} \cdot \frac{6}{7}$

Exercise 2 *(Solution on p. 10.)*

$\frac{a^3}{b^2c^2} \cdot \frac{c^5}{a^5}$

Exercise 3 *(Solution on p. 10.)*

$\frac{y-1}{y^2+1} \cdot \frac{y+1}{y^2-1}$

Exercise 4 *(Solution on p. 10.)*

$\frac{x^2-x-12}{x^2+7x+6} \cdot \frac{x^2-4x-5}{x^2-9x+20}$

Exercise 5 *(Solution on p. 10.)*

$\frac{x^2+6x+8}{x^2-6x+8} \cdot \frac{x^2-2x-8}{x^2+2x-8}$

5 Division Of Rational Expressions

To divide one rational expression by another, we first invert the divisor then multiply the two expressions. Symbolically, if we let P, Q, R, and S represent polynomials, we can write

$\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R} = \frac{P \cdot S}{Q \cdot R}$

6 Sample Set B

Perform the following divisions.

Example 10

$\frac{6x^2}{5a} \div \frac{2x}{10a^3}$.	Invert the divisor and multiply.
$\frac{\overset{3}{\cancel{6}}x^{\cancel{2}}}{\cancel{5}a} \cdot \frac{\overset{2}{\cancel{10}}a^{\cancel{3}}}{\cancel{2}x} = \frac{3x \cdot 2a^2}{1} = 6a^2x$	

Example 11

$\frac{x^2+3x-10}{2x-2} \div \frac{x^2+9x+20}{x^2+3x-4}$	Invert and multiply.
$\frac{x^2+3x-10}{2x-2} \cdot \frac{x^2+3x-4}{x^2+9x+20}$	Factor.
$\frac{\cancel{(x+5)}(x-2)}{2\cancel{(x-1)}} \cdot \frac{\cancel{(x+4)}\cancel{(x-1)}}{\cancel{(x+5)}\cancel{(x+4)}}$	
$\frac{x-2}{2}$	

Example 12

$(4x+7) \div \frac{12x+21}{x-2}$.	Write $4x+7$ as $\frac{4x+7}{1}$.
$\frac{4x+7}{1} \div \frac{12x+21}{x-2}$	Invert and multiply.
$\frac{4x+7}{1} \cdot \frac{x-2}{12x+21}$	Factor.
$\frac{\cancel{4x+7}}{1} \cdot \frac{x-2}{3\cancel{(4x+7)}} = \frac{x-2}{3}$	

7 Practice Set B

Perform each division.

Exercise 6 *(Solution on p. 10.)*

$\frac{8m^2n}{3a^5b^2} \div \frac{2m}{15a^7b^2}$

Exercise 7 *(Solution on p. 10.)*

$\frac{x^2-4}{x^2+x-6} \div \frac{x^2+x-2}{x^2+4x+3}$

Exercise 8 *(Solution on p. 10.)*

$\frac{6a^2+17a+12}{3a+2} \div (2a+3)$

8 Excercises

For the following problems, perform the multiplications and divisions.

Exercise 9 *(Solution on p. 10.)*

$\frac{4a^3}{5b} \cdot \frac{3b}{2a}$

Exercise 10

$\frac{9x^4}{4y^3} \cdot \frac{10y}{x^2}$

Exercise 11 *(Solution on p. 10.)*

$\frac{a}{b} \cdot \frac{b}{a}$

Exercise 12

$\frac{2x}{5y} \cdot \frac{5y}{2x}$

Exercise 13 *(Solution on p. 10.)*

$\frac{12a^3}{7} \cdot \frac{28}{15a}$

Exercise 14

$\frac{39m^4}{16} \cdot \frac{4}{13m^2}$

Exercise 15 *(Solution on p. 10.)*

$\frac{18x^6}{7} \cdot \frac{1}{4x^2}$

Exercise 16

$\frac{34a^6}{21} \cdot \frac{42}{17a^5}$

Exercise 17 *(Solution on p. 10.)*

$\frac{16x^6y^3}{15x^2} \cdot \frac{25x}{4y}$

Exercise 18

$\frac{27a^7b^4}{39b} \cdot \frac{13a^4b^2}{16a^5}$

Exercise 19 *(Solution on p. 10.)*

$\frac{10x^2y^3}{7y^5} \cdot \frac{49y}{15x^6}$

Exercise 20

$\frac{22m^3n^4}{11m^6n} \cdot \frac{33mn}{4mn^3}$

Exercise 21 *(Solution on p. 10.)*

$\frac{-10p^2q}{7a^3b^2} \cdot \frac{21a^5b^3}{2p}$

Exercise 22

$\frac{-25m^4n^3}{14r^3s^3} \cdot \frac{21rs^4}{10mn}$

Exercise 23 *(Solution on p. 10.)*

$\frac{9}{a} \div \frac{3}{a^2}$

Exercise 24

$\frac{10}{b^2} \div \frac{4}{b^3}$

Exercise 25 *(Solution on p. 10.)*

$\frac{21a^4}{5b^2} \div \frac{14a}{15b^3}$

Exercise 26

$\frac{42x^5}{16y^4} \div \frac{21x^4}{8y^3}$

Exercise 27 *(Solution on p. 10.)*

$\frac{39x^2y^2}{55p^2} \div \frac{13x^3y}{15p^6}$

Exercise 28

$\frac{14mn^3}{25n^6} \div \frac{32m}{20m^2n^3}$

Exercise 29 *(Solution on p. 10.)*

$\frac{12a^2b^3}{-5xy^4} \div \frac{6a^2}{15x^2}$

Exercise 30
$\frac{24p^3q}{9mn^3} \div \frac{10pq}{-21n^2}$

Exercise 31 *(Solution on p. 10.)*
$\frac{x+8}{x+1} \cdot \frac{x+2}{x+8}$

Exercise 32
$\frac{x+10}{x-4} \cdot \frac{x-4}{x-1}$

Exercise 33 *(Solution on p. 10.)*
$\frac{2x+5}{x+8} \cdot \frac{x+8}{x-2}$

Exercise 34
$\frac{y+2}{2y-1} \cdot \frac{2y-1}{y-2}$

Exercise 35 *(Solution on p. 10.)*
$\frac{x-5}{x-1} \div \frac{x-5}{4}$

Exercise 36
$\frac{x}{x-4} \div \frac{2x}{5x+1}$

Exercise 37 *(Solution on p. 10.)*
$\frac{a+2b}{a-1} \div \frac{4a+8b}{3a-3}$

Exercise 38
$\frac{6m+2}{m-1} \div \frac{4m-4}{m-1}$

Exercise 39 *(Solution on p. 10.)*
$x^3 \cdot \frac{4ab}{x}$

Exercise 40
$y^4 \cdot \frac{3x^2}{y^2}$

Exercise 41 *(Solution on p. 11.)*
$2a^5 \div \frac{6a^2}{4b}$

Exercise 42
$16x^2y^3 \div \frac{10xy}{3}$

Exercise 43 *(Solution on p. 11.)*
$21m^4n^2 \div \frac{3mn^2}{7n}$

Exercise 44
$(x+8) \cdot \frac{x+2}{x+8}$

Exercise 45 *(Solution on p. 11.)*
$(x-2) \cdot \frac{x-1}{x-2}$

Exercise 46
$(a-6)^3 \cdot \frac{(a+2)^2}{a-6}$

Exercise 47 *(Solution on p. 11.)*
$(b+1)^4 \cdot \frac{(b-7)^3}{b+1}$

Exercise 48
$\left(b^2+2\right)^3 \cdot \frac{b-3}{(b^2+2)^2}$

Exercise 49 *(Solution on p. 11.)*
$\left(x^3-7\right)^4 \cdot \frac{x^2-1}{(x^3-7)^2}$

Exercise 50
$(x-5) \div \frac{x-5}{x-2}$

Exercise 51 *(Solution on p. 11.)*
$(y-2) \div \frac{y-2}{y-1}$

Exercise 52

$(y+6)^3 \div \frac{(y+6)^2}{y-6}$

Exercise 53 *(Solution on p. 11.)*

$(a-2b)^4 \div \frac{(a-2b)^2}{a+b}$

Exercise 54

$\frac{x^2+3x+2}{x^2-4x+3} \cdot \frac{x^2-2x-3}{2x+2}$

Exercise 55 *(Solution on p. 11.)*

$\frac{6x-42}{x^2-2x-3} \cdot \frac{x^2-1}{x-7}$

Exercise 56

$\frac{3a+3b}{a^2-4a-5} \div \frac{9a+9b}{a^2-3a-10}$

Exercise 57 *(Solution on p. 11.)*

$\frac{a^2-4a-12}{a^2-9} \div \frac{a^2-5a-6}{a^2+6a+9}$

Exercise 58

$\frac{b^2-5b+6}{b^2-b-2} \cdot \frac{b^2-2b-3}{b^2-9b+20}$

Exercise 59 *(Solution on p. 11.)*

$\frac{m^2-4m+3}{m^2+5m-6} \cdot \frac{m^2+4m-12}{m^2-5m+6}$

Exercise 60

$\frac{r^2+7r+10}{r^2-2r-8} \div \frac{r^2+6r+5}{r^2-3r-4}$

Exercise 61 *(Solution on p. 11.)*

$\frac{2a^2+7a+3}{3a^2-5a-2} \cdot \frac{a^2-5a+6}{a^2+2a-3}$

Exercise 62

$\frac{6x^2+x-2}{2x^2+7x-4} \cdot \frac{x^2+2x-12}{3x^2-4x-4}$

Exercise 63 *(Solution on p. 11.)*

$\frac{x^3y-x^2y^2}{x^2y-y^2} \cdot \frac{x^2-y}{x-xy}$

Exercise 64

$\frac{4a^3b-4a^2b^2}{15a-10} \cdot \frac{3a-2}{4ab-2b^2}$

Exercise 65 *(Solution on p. 11.)*

$\frac{x+3}{x-4} \cdot \frac{x-4}{x+1} \cdot \frac{x-2}{x+3}$

Exercise 66

$\frac{x-7}{x+8} \cdot \frac{x+1}{x-7} \cdot \frac{x+8}{x-2}$

Exercise 67 *(Solution on p. 11.)*

$\frac{2a-b}{a+b} \cdot \frac{a+3b}{a-5b} \cdot \frac{a-5b}{2a-b}$

Exercise 68

$\frac{3a(a+1)^2}{a-5} \cdot \frac{6(a-5)^2}{5a+5} \cdot \frac{15a+30}{4a-20}$

Exercise 69 *(Solution on p. 11.)*

$\frac{-3a^2}{4b} \cdot \frac{-8b^3}{15a}$

Exercise 70

$\frac{-6x^3}{5y^2} \cdot \frac{20y}{-2x}$

Exercise 71 *(Solution on p. 11.)*

$\frac{-8x^2y^3}{-5x} \div \frac{4}{-15xy}$

Exercise 72

$\frac{-4a^3}{3b} \div \frac{2a}{6b^2}$

Exercise 73 *(Solution on p. 11.)*

$\frac{-3a-3}{2a+2} \cdot \frac{a^2-3a+2}{a^2-5a-6}$

Exercise 74
$\frac{x^2-x-2}{x^2-3x-4} \cdot \frac{-x^2+2x+3}{-4x-8}$

Exercise 75 *(Solution on p. 11.)*
$\frac{-5x-10}{x^2-4x+3} \cdot \frac{x^2+4x+1}{x^2+x-2}$

Exercise 76
$\frac{-a^2-2a+15}{-6a-12} \div \frac{a^2-2a-8}{-2a-10}$

Exercise 77 *(Solution on p. 11.)*
$\frac{-b^2-5b+14}{3b-6} \div \frac{-b^2-9b-14}{-b+8}$

Exercise 78
$\frac{3a+6}{4a-24} \cdot \frac{6-a}{3a+15}$

Exercise 79 *(Solution on p. 11.)*
$\frac{4x+12}{x-7} \cdot \frac{7-x}{2x+2}$

Exercise 80
$\frac{-2b-2}{b^2+b-6} \cdot \frac{-b+2}{b+5}$

Exercise 81 *(Solution on p. 11.)*
$\frac{3x^2-6x-9}{2x^2-6x-4} \div \frac{3x^2-5x-2}{6x^2-7x-3}$

Exercise 82
$\frac{-2b^2-2b+4}{8b^2-28b-16} \div \frac{b^2-2b+1}{2b^2-5b-3}$

Exercise 83 *(Solution on p. 11.)*
$\frac{x^2+4x+3}{x^2+5x+4} \div (x+3)$

Exercise 84
$\frac{x^2-3x+2}{x^2-4x+3} \div (x-3)$

Exercise 85 *(Solution on p. 11.)*
$\frac{3x^2-21x+18}{x^2+5x+6} \div (x+2)$

9 Exercises For Review

Exercise 86
(here[1]) If $a < 0$, then $|a| =$_____.

Exercise 87 *(Solution on p. 12.)*
(here[2]) Classify the polynomial $4xy + 2y$ as a monomial, binomial, or trinomial. State its degree and write the numerical coefficient of each term.

Exercise 88
(here[3]) Find the product: $y^2 (2y-1)(2y+1)$.

Exercise 89 *(Solution on p. 12.)*
(here[4]) Translate the sentence "four less than twice some number is two more than the number" into an equation.

[1] "Basic Operations with Real Numbers: Absolute Value" <http://cnx.org/content/m21876/latest/>
[2] "Algebraic Expressions and Equations: Classification of Expressions and Equations" <http://cnx.org/content/m21848/latest/>
[3] "Algebraic Expressions and Equations: Combining Polynomials Using Addition and Subtraction" <http://cnx.org/content/m21854/latest/>
[4] "Solving Linear Equations and Inequalities: Application I - Translating from Verbal to Mathetical Expressions" <http://cnx.org/content/m21978/latest/>

Exercise 90
(here[5]**)** Reduce the fraction $\frac{x^2-4x+4}{x^2-4}$.

[5] "Rational Expressions: Reducing Rational Expressions" <http://cnx.org/content/m21953/latest/>

Solutions to Exercises in this Module

Solution to Exercise (p. 3)
$\frac{10}{7}$
Solution to Exercise (p. 3)
$\frac{c^3}{a^2b^2}$
Solution to Exercise (p. 3)
$\frac{1}{y^2+1}$
Solution to Exercise (p. 3)
$\frac{x+3}{x+6}$
Solution to Exercise (p. 3)
$\frac{(x+2)^2}{(x-2)^2}$
Solution to Exercise (p. 4)
$20a^2mn$
Solution to Exercise (p. 4)
$\frac{x+1}{x-1}$
Solution to Exercise (p. 4)
$\frac{3a+4}{3a+2}$
Solution to Exercise (p. 5)
$\frac{6a^2}{5}$
Solution to Exercise (p. 5)
1
Solution to Exercise (p. 5)
$\frac{16a^2}{5}$
Solution to Exercise (p. 5)
$\frac{9x^4}{14}$
Solution to Exercise (p. 5)
$\frac{20x^5y^2}{3}$
Solution to Exercise (p. 5)
$\frac{14}{3x^4y}$
Solution to Exercise (p. 5)
$-15a^2bpq$
Solution to Exercise (p. 5)
$3a$
Solution to Exercise (p. 5)
$\frac{9a^3b}{2}$
Solution to Exercise (p. 5)
$\frac{9p^4y}{11x}$
Solution to Exercise (p. 5)
$\frac{-6b^3x}{y^4}$
Solution to Exercise (p. 6)
$\frac{x+2}{x+1}$
Solution to Exercise (p. 6)
$\frac{2x+5}{x-2}$
Solution to Exercise (p. 6)
$\frac{4}{x-1}$
Solution to Exercise (p. 6)
$\frac{3}{4}$

Solution to Exercise (p. 6)
$4abx^2$
Solution to Exercise (p. 6)
$\frac{4a^3b}{3}$
Solution to Exercise (p. 6)
$49m^3n$
Solution to Exercise (p. 6)
$x-1$
Solution to Exercise (p. 6)
$(b+1)^3(b-7)^3$
Solution to Exercise (p. 6)
$\left(x^3-7\right)^2(x+1)(x-1)$
Solution to Exercise (p. 6)
$(y-1)$
Solution to Exercise (p. 7)
$(a-2b)^2(a+b)$
Solution to Exercise (p. 7)
$\frac{6(x-1)}{(x-3)}$
Solution to Exercise (p. 7)
$\frac{(a+2)(a+3)}{(a-3)(a+1)}$
Solution to Exercise (p. 7)
1
Solution to Exercise (p. 7)
$\frac{(2a+1)(a-6)(a+1)}{(3a+1)(a-1)(a-2)}$
Solution to Exercise (p. 7)
$\frac{x(x-y)}{1-y}$
Solution to Exercise (p. 7)
$\frac{x-2}{x+1}$
Solution to Exercise (p. 7)
$\frac{a+3b}{a+b}$
Solution to Exercise (p. 7)
$\frac{2ab^2}{5}$
Solution to Exercise (p. 7)
$-6x^2y^4$
Solution to Exercise (p. 7)
$\frac{-3(a-2)(a-1)}{2(a-6)(a+1)}$
Solution to Exercise (p. 8)
$\frac{-5\left(x^2+4x+1\right)}{(x-3)(x-1)^2}$
Solution to Exercise (p. 8)
$\frac{-(b-8)}{3(b+2)}$
Solution to Exercise (p. 8)
$\frac{-2(x+3)}{(x+1)}$
Solution to Exercise (p. 8)
$\frac{3(x-3)(x+1)(2x-3)}{2(x^2-3x-2)(x-2)}$
Solution to Exercise (p. 8)
$\frac{(x+4)(x-1)}{(x+3)(x^2-4x-3)}$

Solution to Exercise (p. 8)

$\frac{3(x-6)(x-1)}{(x+2)^2(x+3)}$

Solution to Exercise (p. 8)

binomial; 2; 4, 2

Solution to Exercise (p. 8)

$2x - 4 = x + 2$